Optimizing quantum violation for multipartite facet Bell inequalities

Jinfu Chen | @ NWO Physics, 20.1.2026

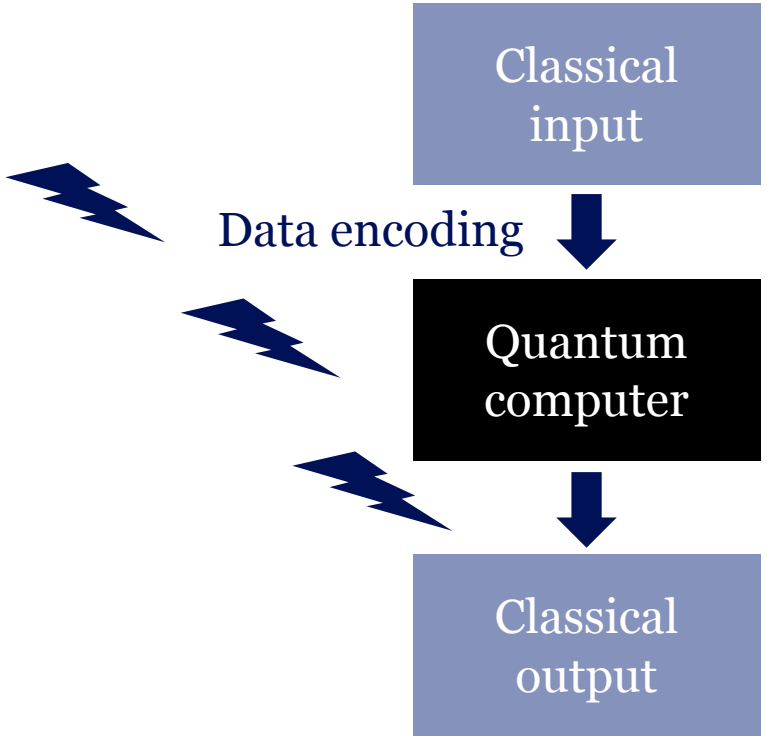
Universiteit Leiden
The Netherlands

arXiv:2511.07523 with **Mengyao Hu, Jordi Tura**

Instituut-Lorentz
Leiden Institute of Physics
Applied Quantum Algorithms
erc

Quantum computer

Applied quantum algorithms:



$x, y...$

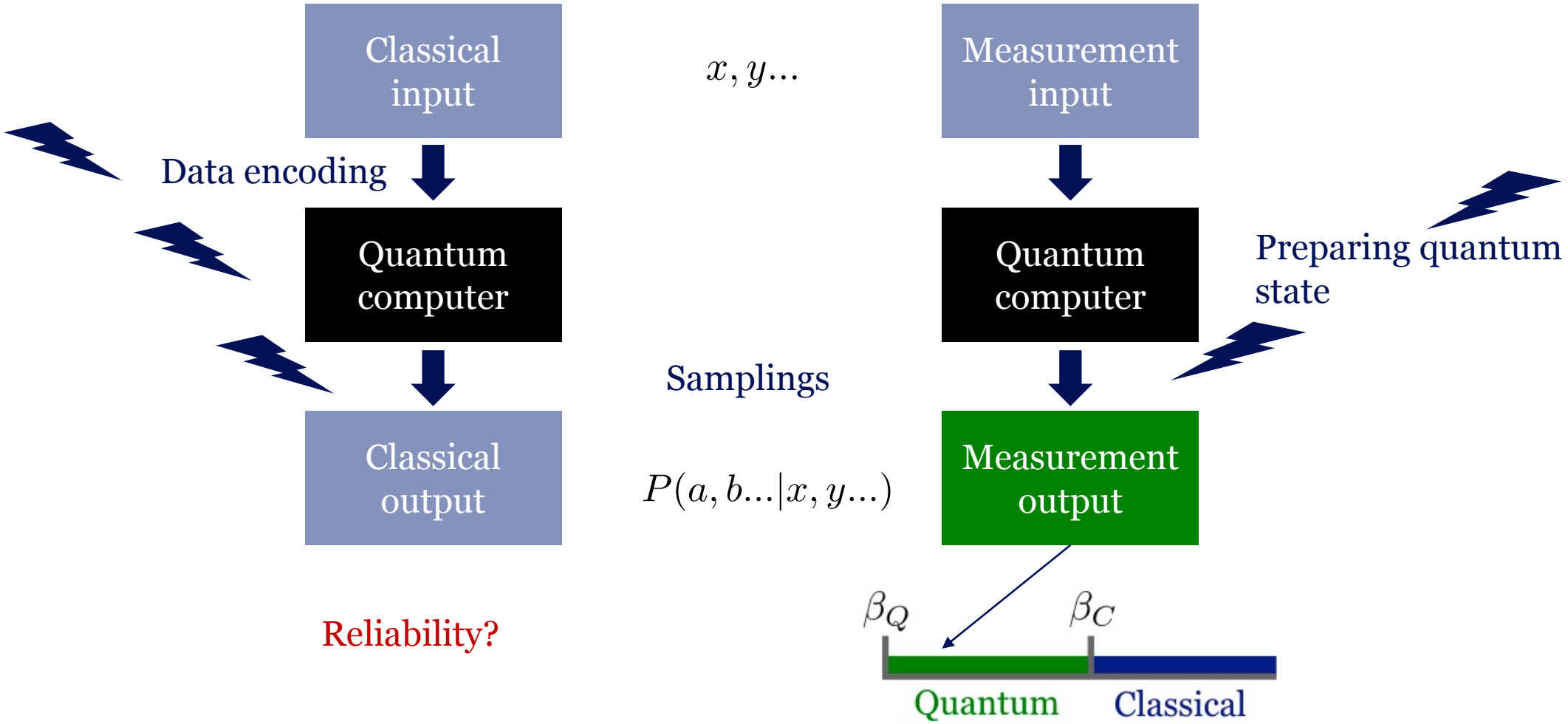
Samplings

$P(a, b...|x, y...)$

Reliability?

Applied quantum algorithms:

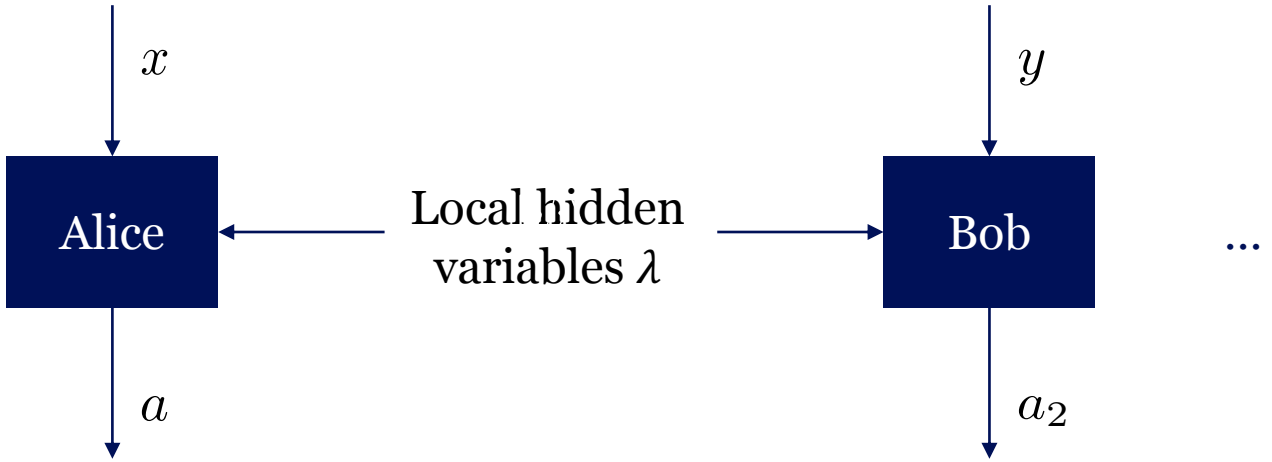
Quantum certification:



A crash course on nonlocality

- Input $x, y = 1,2, \dots, m$: choose different measurements
- Outcome $a, b = 1,2, \dots, d$: the result of the measurements

$$P(a, b|x, y) := P(\boldsymbol{a}|\boldsymbol{x})$$

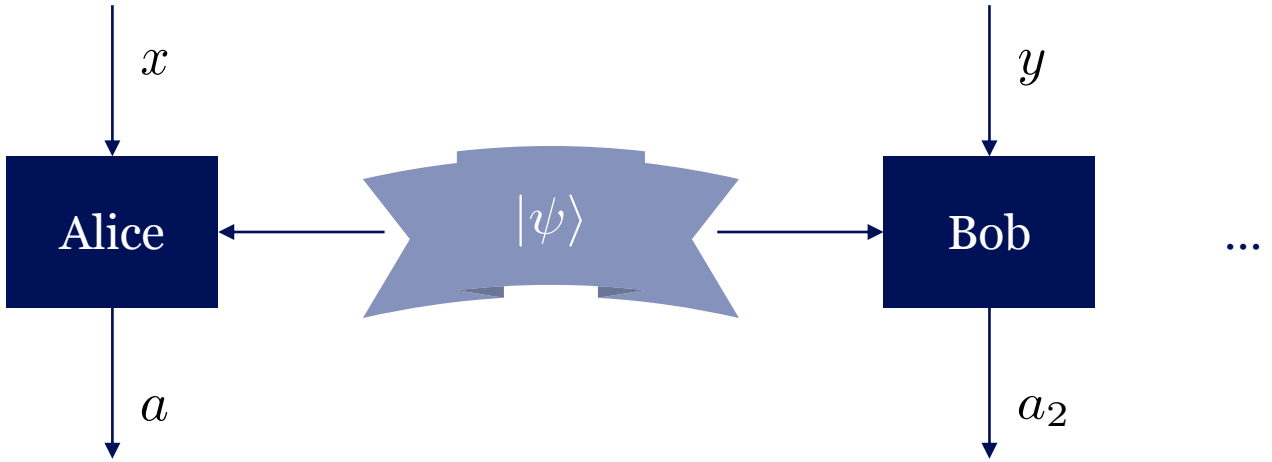


- Probability: $P(\boldsymbol{a}|\boldsymbol{x}) \geq 0, \sum_{\boldsymbol{a}} P(\boldsymbol{a}|\boldsymbol{x}) = 1$

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Bell inequalities

- Linear functional in probability space

$$I = \sum_{\boldsymbol{a},\boldsymbol{x}} T_{\boldsymbol{a},\boldsymbol{x}} P(\boldsymbol{a}|\boldsymbol{x})$$

- Classical bound: min in **Local polytope** $\mathcal{L}$

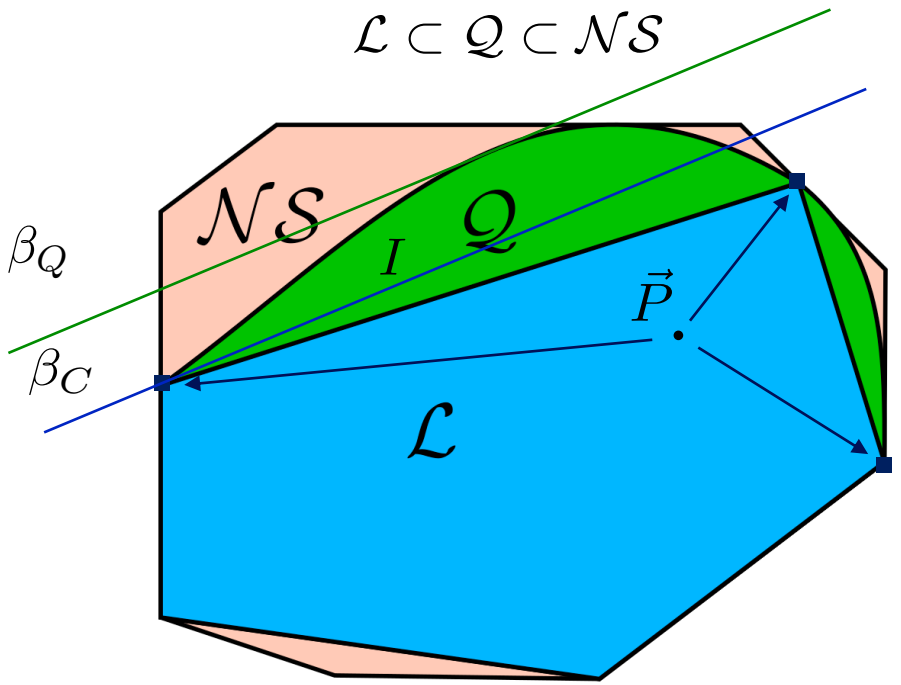
$$\beta_C = \min_{\vec{P}\in\mathcal{L}} I$$

- Quantum value: min in **Quantum set** $\mathcal{Q}$

$$\beta_Q = \min_{\vec{P}\in\mathcal{Q}} I = \min_{\boldsymbol{\theta},|\psi\rangle} \langle\psi|\, \hat{I}(\boldsymbol{\theta})\, |\psi\rangle$$

- Bell operator

$$\hat{I}(\boldsymbol{\theta}) = \sum_{\boldsymbol{a},\boldsymbol{x}} T_{\boldsymbol{a},\boldsymbol{x}} \hat{M}_{\boldsymbol{a}|\boldsymbol{x}}(\boldsymbol{\theta})$$



Non-Signaling polytope $\mathcal{NS}$

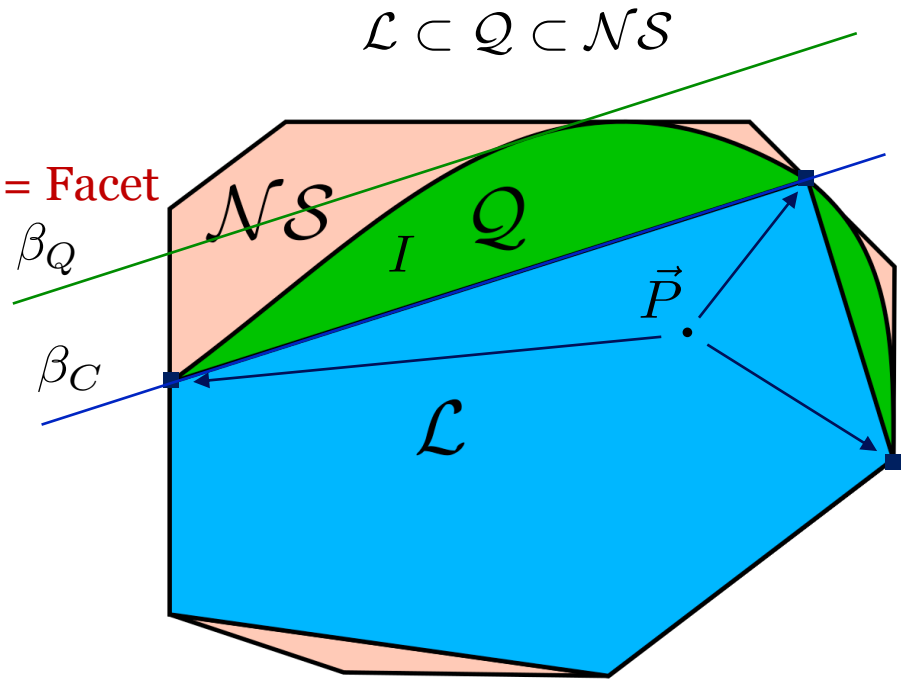
$$\sum_{b=1}^{d} P(a,b|x,y) = \sum_{b=1}^{d} P(a,b|x,y')$$

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Tight Bell inequality = Facet



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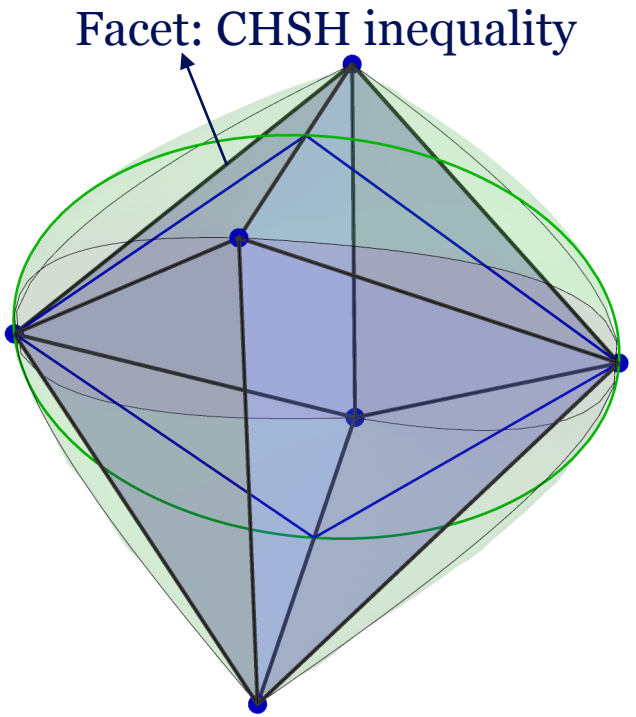
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2 parties, 2 measurements, 2 outcomes



Local polytope

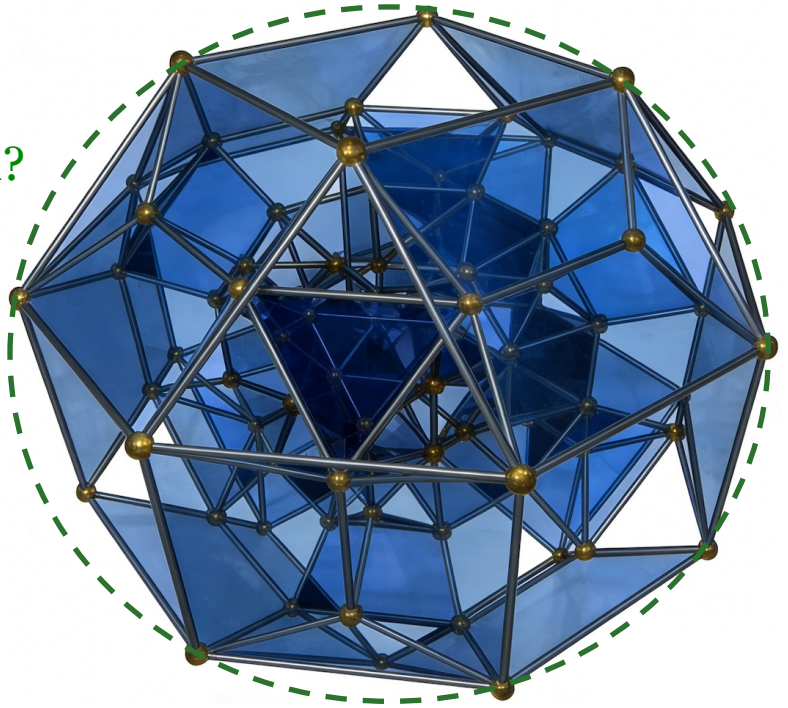
Quantum set

Bell correlation in many-body regime

- Multipartite Bell correlation are harder: $P(a, b, c, ...|x, y, z...)$

  - Exponential curse in many-body physics ➡ Handle by adding symmetry or locality constraint

- Robust multipartite Bell inequalities? Quantum violation?

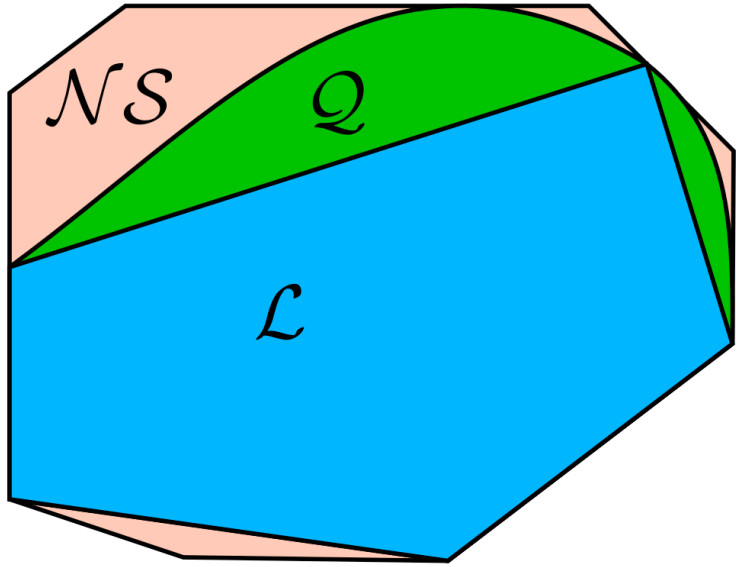


- Useful in
  - Certify your quantum computer
  - Genuine multipartite entanglement
  - Collective advantage in metrology
  - Quantum network and distributed quantum computing

Bell correlation in many-body regime

- N party, m measurements, d outcomes: (N, m, d) scenario

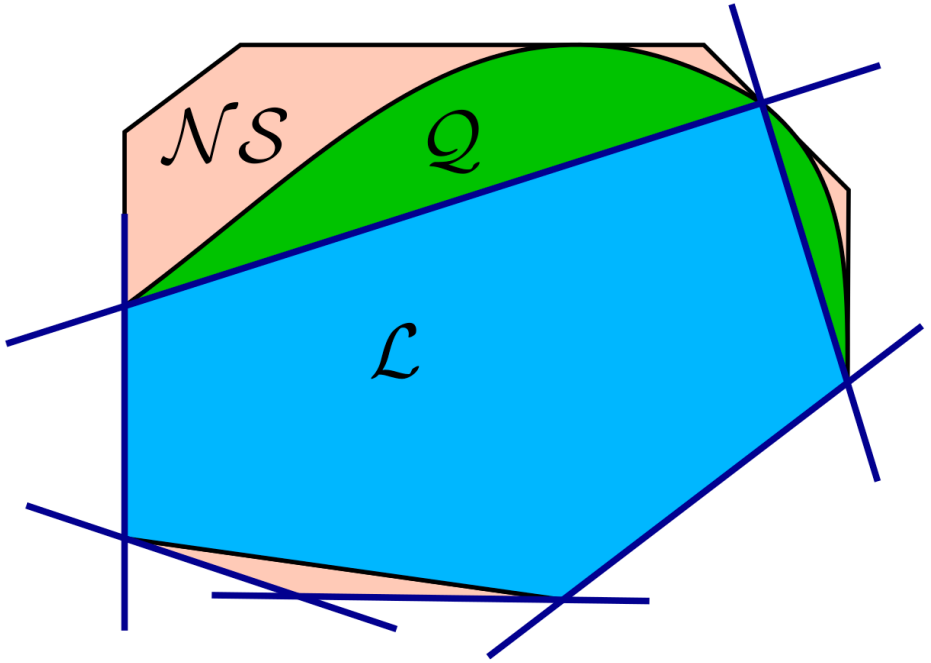
  Finding all tight Bell inequalities



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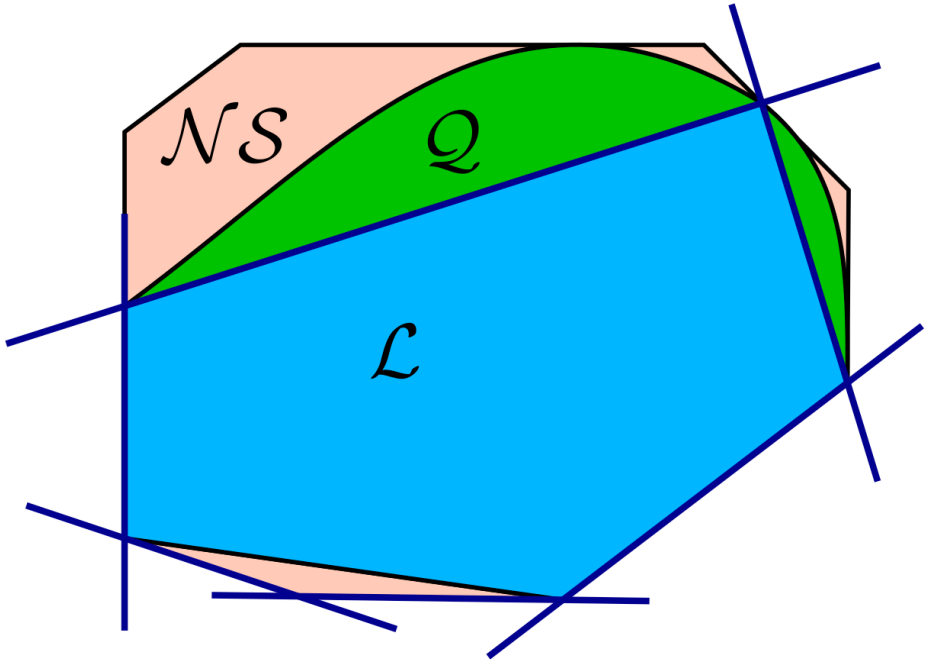
Finding all tight Bell inequalities ⟷ All facets of local polytope



Bell correlation in many-body regime

- N party, m measurements, d outcomes: (N, m, d) scenario

Finding all tight Bell inequalities $\longleftrightarrow$ All facets of local polytope



- Brute searching is not realistic:

Complexity: $O(v^{\lfloor D/2 \rfloor} + v \log v)$

Number of vertices: $v = d^{mN}$

Dimension of local polytope: $D = (md)^N$

$(2, 2, 2) \longrightarrow < 1\text{ms}$

$(3, 2, 2) \longrightarrow 5$ seconds

$(4, 2, 2) \longrightarrow 10^{67}$ years

$\vdots$

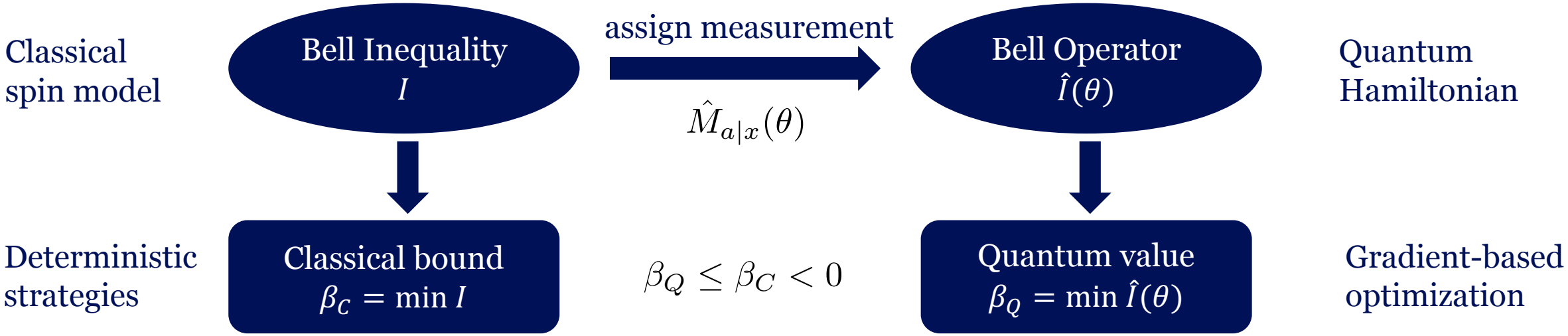
$(10^4, 2, 2) \longrightarrow 10^{10^{4.67867\ldots}}$ (basically any timescale you want)

Tackle the bottleneck

- Reduce by symmetry: Permutation invariant, Translation invariant
  - More realistic for many-body experiments
  - Large N: cold atoms N. J. Engelsen, R. Krishnakumar, O. Hosten, and M. A. Kasevich, Phys. Rev. Lett. 118, 140401 (2017)
  - Small N : photonics, superconducting qubits.

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J. Tura, G. De las Cuevas, R. Augusiak, M. Lewenstein, A. Acín, and J. I. Cirac,
Energy as a Detector of Nonlocality of Many-Body Spin Systems, Phys. Rev. X 7, 021005 (2017)

Tackle the bottleneck

- Certification typically requires only some noise-tolerant inequalities
- No need to enumerate all tight Bell inequalities

Finding more robust Bell inequality:

$$\text{fix } \beta_C,\ \beta_Q \downarrow$$

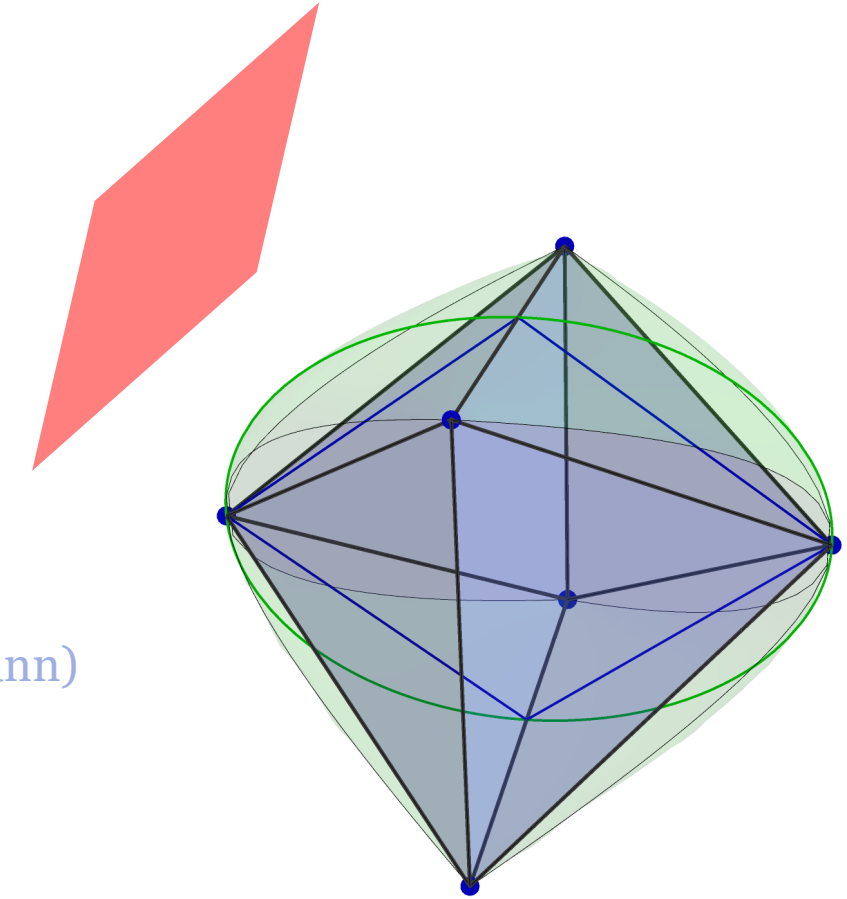


Enhancing quantum-classical ratio

$$\Delta = \beta_Q / \beta_C$$

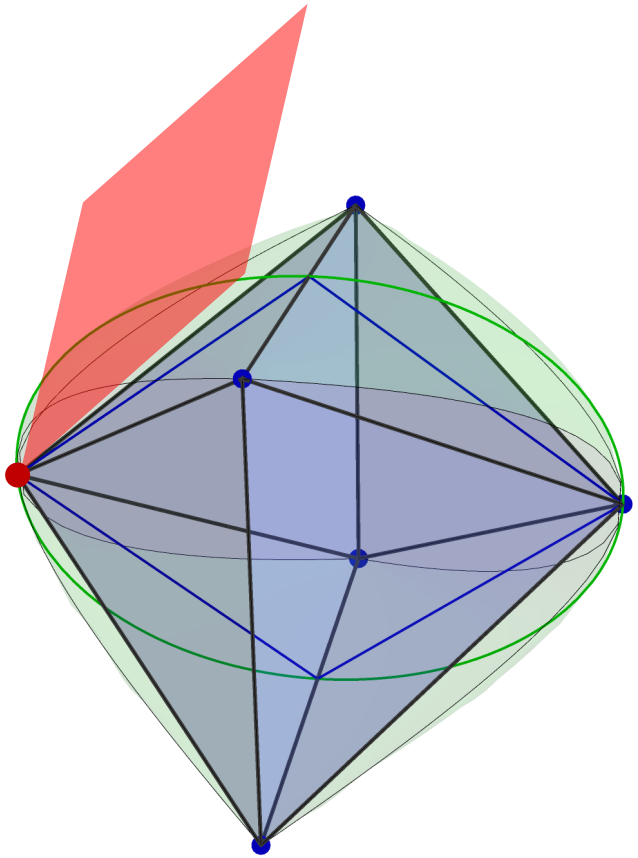
Optimization procedure

- Start from a linear functional (not tight)
- Find vertices on local polytope saturating the classical bound β_C
- Fix them: they constrain α_{ij} and determine β_C
- Decrease β_Q under these constraints (gradient via Feynman–Hellmann)
- If new vertices reach β_C, add them → tighter inequality
- Iterate until: facet or gradient ≈ 0



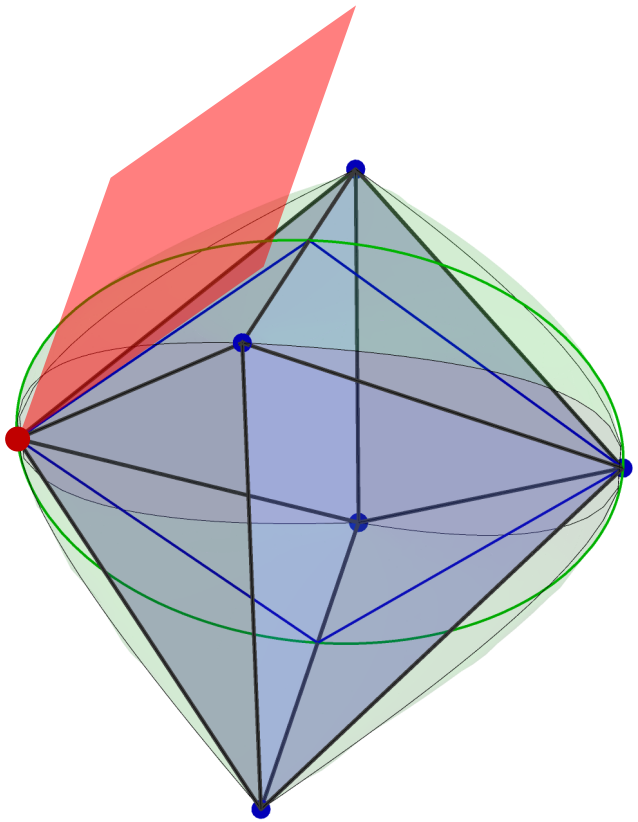
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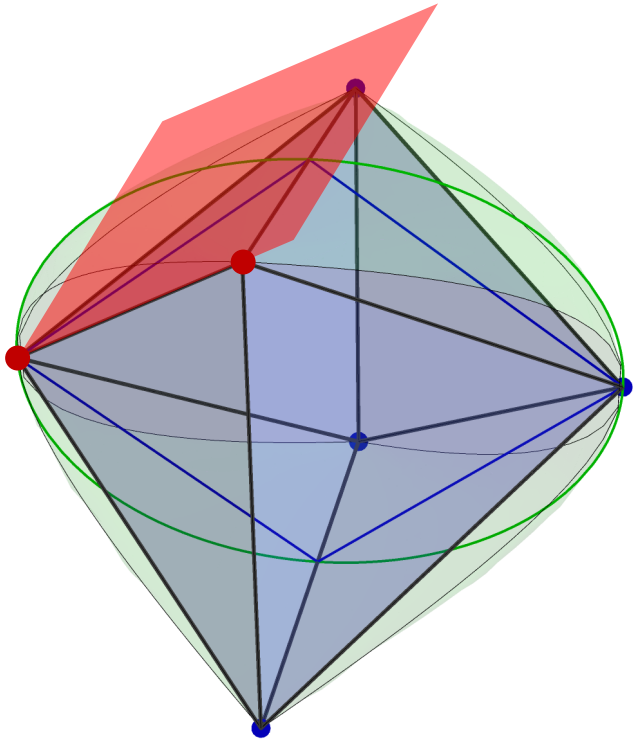
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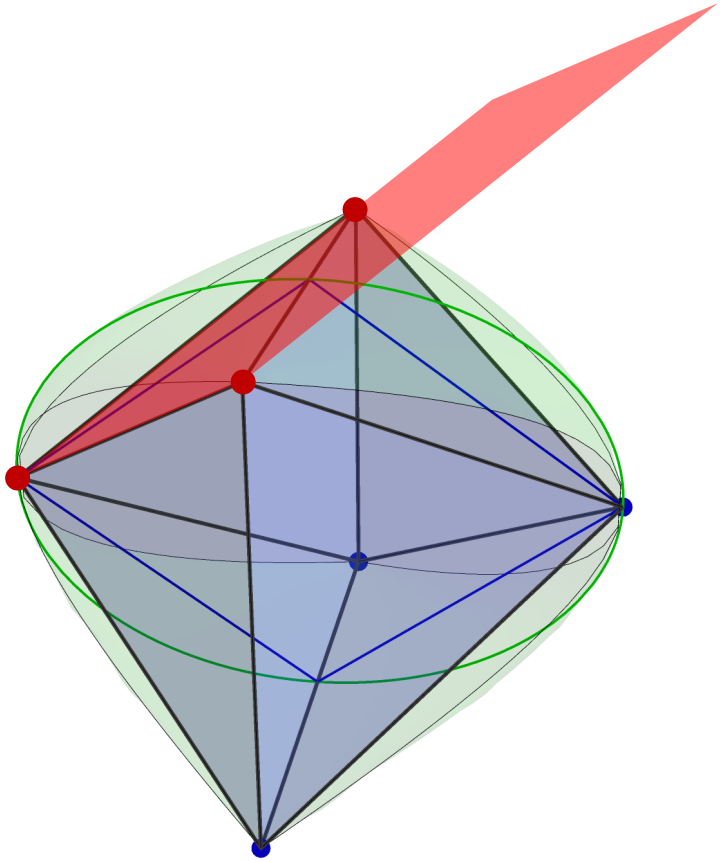
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Benchmark: Bipartite scenario

- $(N = 2, m = 2, d = 2)$ scenario

$$I = \alpha_{00} \langle A_0 B_0 \rangle + \alpha_{01} \langle A_0 B_1 \rangle + \alpha_{10} \langle A_1 B_0 \rangle + \alpha_{11} \langle A_1 B_1 \rangle$$

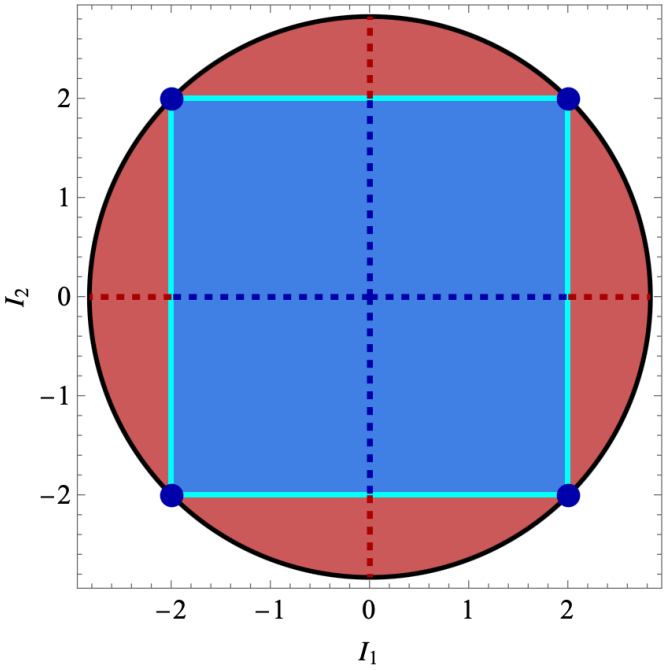
- Classical bound: the minimum among 16 choices

$$\beta_C = \min_{a_i, b_j = \pm 1} (\alpha_{00} a_0 b_0 + \alpha_{01} a_0 b_1 + \alpha_{10} a_1 b_0 + \alpha_{11} a_1 b_1)$$

- Quantum value: (1) ground-state energy

  (2) optimizing measurement strategy

$$\beta_Q = \min_{\boldsymbol{\theta}, |\psi\rangle} \hat{I}(\boldsymbol{\theta})$$



$$I_1 = \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle$$

$$I_2 = -\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle + \langle A_1 B_1 \rangle$$

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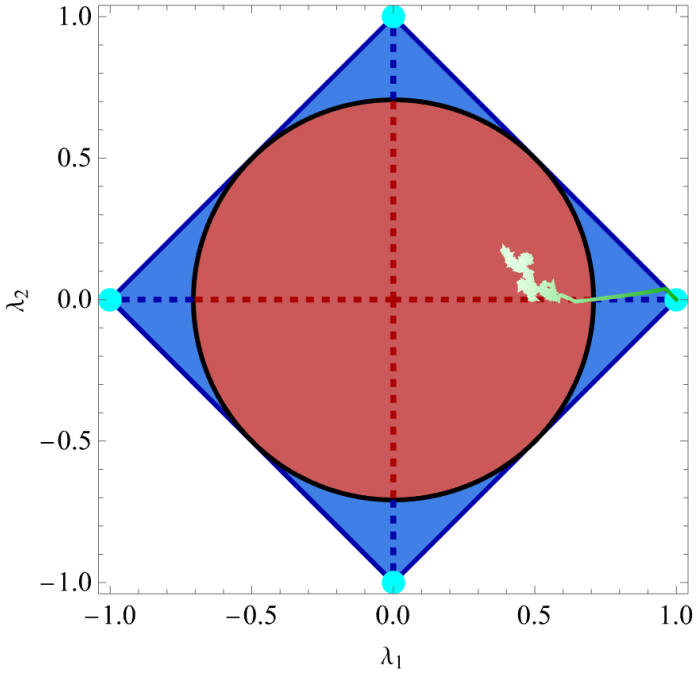
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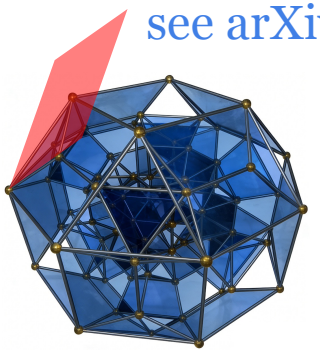
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Summary and outlook

Summary

- **General gradient-based method** for optimizing Bell inequalities
- **Validation** on the (2,2,2) scenario (recovers known tight inequalities)
- **Application** to permutation-invariant Bell inequalities
- **Analytical results** in the thermodynamic limit $N \to \infty$

see arXiv:2511.07523



Future work

- **Certified lower bounds** on the optimal ratio β_Q
- Extensions to new scenarios:
  - Translation-invariant settings
  - Bipartite cases with larger m (inputs) and d (outcomes)
  - General non-symmetric scenarios whenever **enumerating all facet is infeasible**

Thank you!

- Collaborators:



Mengyao Hu



Jordi Tura

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