

Boosting thermalization of classical and quantum many-body systems

Jinfu Chen | @StatPhys29, 14.7.2025

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arXiv:2411.03420 & ongoing work with **Kshiti Sneh Rai, Patrick Emonts, Donato Farina, Marcin Płodzień, Przemyslaw Grzybowski, Maciej Lewenstein, Jordi Tura**

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Motivation: thermal state preparation

- Stochastic thermodynamics ➡ Open quantum systems

Classical detailed balance condition
Continuous-time Markov chain

$$\dot{p} = Lp$$

Quantum detailed balance condition
Lindblad master equation (Lindbladian)

$$\dot{\rho} = \mathcal{L}[\rho]$$

- **Goal**: Prepare classical & quantum thermal states $\rho_\beta = e^{-\beta h}/Z(\beta)$
  - A framework to **design kinetic rules** L or $\mathcal{L}[\cdot]$ → Unique steady state
  - A method to **optimize kinetic coeffcients** → Fast thermalization

F. Avanzini, M. Bilancioni, V. Cavina, S. Dal Cengio, M. Esposito, G. Falasco, *et al. SciPost Phys. Lect. Notes* 80 (2024).

Motivation: boosting thermalization

- Thermalization dynamics

$$\dot{\rho} = \mathcal{L}[\rho]$$

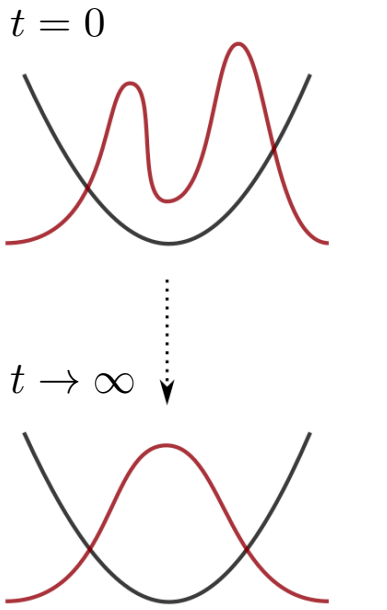
$$\rho(t) = \rho_\beta + \rho_1 e^{\lambda_1 t} + \rho_2 e^{\lambda_2 t} + \cdots$$

Slowest relaxing mode

$$\lambda_0 = 0$$

- Spectral gap of Lindbladian

$$\Delta = -\mathrm{Re}(\lambda_1) \leq -\mathrm{Re}(\lambda_2) \leq \cdots$$



Harmonic potential

Maximizing Δ ➡ Fast thermalization ➡ Reducing irreversibility

- For many-body systems, $\mathcal{L}$ has too many free parameters to determine.

Content

- Preparing classical thermal states
  - A single two-level system
  - Kinetic Ising model
  - Spectral gap optimization
- Preparing quantum thermal states
  - Constructing Lindbladians under quantum detailed balance
  - Lipkin-Meshkov-Glick model
  - Transverse Ising model
- Summary

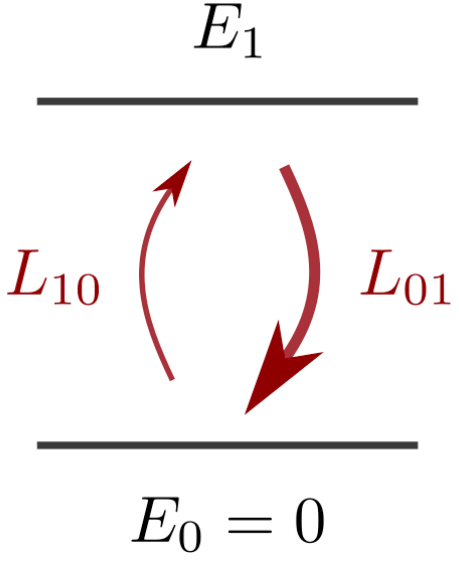
Preparing classical thermal states

- Two-level system (a bit)

$$h^{\mathrm{cl}} = \begin{pmatrix} 0 & \\ & E_1 \end{pmatrix}$$

$$p_1^{\mathrm{eq}} = \frac{e^{-\beta E_1}}{1+e^{-\beta E_1}}$$

$$p_0^{\mathrm{eq}} = \frac{1}{1+e^{-\beta E_1}}$$



- Liouvillian

$$L = \begin{pmatrix} -\Gamma e^{-\beta E_1/2} & \Gamma e^{\beta E_1/2} \\ \Gamma e^{-\beta E_1/2} & -\Gamma e^{\beta E_1/2} \end{pmatrix}$$

dissipation jump

- Detailed-balanced condition:

$$\frac{L_{01}}{L_{10}} = e^{\beta E_1}$$

Parent Hamiltonian

- **Detailed balance condition** allows symmetrizing L into a Hermitian operator

$$H = -e^{\frac{\beta}{2}h^{\mathrm{cl}}} L e^{-\frac{\beta}{2}h^{\mathrm{cl}}} = \Gamma \begin{pmatrix} e^{-\frac{\beta}{2}E_1} & -1 \\ -1 & e^{\frac{\beta}{2}E_1} \end{pmatrix}$$

- **Similarity transformation** preserves the spectrum
- H is Hermitian — called the ***parent Hamiltonian***

- **Two-level system**: ratio of transition rates is fixed by detailed balance
- **Many-body systems**: kinetic coefficients include many free parameters

Kinetic Ising model: Single-spin flip

- Classical spin dynamics under stochastic flips
- Goal: **construct Liouvillians to prepare classical thermal states**
- Infinite temperature $T = \infty \rightarrow$ maximally mixed state $\quad p_{\beta=0}(\boldsymbol{\sigma}) = 1/2^N$
- Use local **single-spin flip operators**:

$$\tilde{F}_i = \sigma_i^x$$

- Liouvillian and Parent Hamiltonian are

$$\tilde{L} = -\tilde{H} = \sum_{i=1}^{N} \tilde{\gamma}_i (\sigma_i^x - \mathbb{1})$$

Kinetic Ising model: Single-spin flip

- Finite-temperature thermal state

$$p_\beta(\boldsymbol{\sigma}) = \frac{e^{-\beta h^{\mathrm{cl}}(\boldsymbol{\sigma})}}{Z(\beta)}, \qquad h^{\mathrm{cl}}(\boldsymbol{\sigma}) = \sum_i -J\sigma_i^z\sigma_{i+1}^z$$

- Detailed balance condition:

$$\frac{L_{\boldsymbol{\sigma}'\leftarrow\boldsymbol{\sigma}}}{L_{\boldsymbol{\sigma}\leftarrow\boldsymbol{\sigma}'}} = \exp\left[\beta(h^{\mathrm{cl}}(\boldsymbol{\sigma}) - h^{\mathrm{cl}}(\boldsymbol{\sigma}'))\right]$$

- Finite-temperature flip operator F: $\langle\boldsymbol{\sigma}'|\, F\, |\boldsymbol{\sigma}\rangle = L_{\boldsymbol{\sigma}'\leftarrow\boldsymbol{\sigma}}$

$$\langle\boldsymbol{\sigma}'|\, F\, |\boldsymbol{\sigma}\rangle = \langle\boldsymbol{\sigma}|\, e^{\beta h^{\mathrm{cl}}} F e^{-\beta h^{\mathrm{cl}}}\, |\boldsymbol{\sigma}'\rangle \quad \Rightarrow \quad F^T = e^{\beta h^{\mathrm{cl}}} F e^{-\beta h^{\mathrm{cl}}}$$

- This implies $e^{\frac{\beta h^{\mathrm{cl}}}{2}} F e^{-\frac{\beta h^{\mathrm{cl}}}{2}} = \tilde{F}$ is symmetric.

Kinetic Ising model: Single-spin flip

- Single-spin flip operator: $\mathscr{A} = \{F_i,\ 1 \le i \le N\}, \qquad F_i = e^{-\frac{\beta h^{\rm cl}}{2}} \tilde{F}_i e^{\frac{\beta h^{\rm cl}}{2}}$
- Liouvillian

$$L = \sum_{i=1}^{N} L_i = \sum_{i=1}^{N} \tilde{\gamma}_i (e^{-\frac{\beta h^{\rm cl}}{2}} \tilde{F}_i e^{\frac{\beta h^{\rm cl}}{2}} - \sigma_i^x e^{-\frac{\beta h^{\rm cl}}{2}} \tilde{F}_i e^{\frac{\beta h^{\rm cl}}{2}})$$

Dissipation: diagonal operator

- Parent Hamiltonian

$$H = \sum_{i=1}^{N} \tilde{\gamma}_i (e^{\frac{\beta h^{\rm cl}}{2}} \sigma_i^x e^{-\frac{\beta h^{\rm cl}}{2}} \tilde{F}_i - \tilde{F}_i)$$

Detailed balance condition

$$L_i^T = e^{\beta h^{\rm cl}} L_i e^{-\beta h^{\rm cl}}$$
$$H^\dagger = H^T = H$$

- For 1D case with $\tilde{F}_i = \sigma_i^x$, F_i acts on sites $i-1, i, i+1$
- Other choices of single-spin flip operators: $\tilde{F}_{in} = (1 \pm \sigma_{i-1}^z \sigma_{i+1}^z)\sigma_i^x/2$
- Optimize free parameters $\tilde{\gamma}_{in}$ for to boost thermalization

Lindbladian and Parent Hamiltonian

	Liouvillian $L = \sum_{i=1}^{N} L_i$	**Parent Hamiltonian** $H = \sum_{i=1}^{N} H_i$
Steady state/ ground state	$\lvert p_\beta \rangle = \frac{e^{-\beta h^{\mathrm{cl}}}}{Z(\beta)} \lvert \mathbf{1} \rangle$ $L \lvert p_\beta \rangle = 0$ $\langle \mathbf{1} \rvert L = 0$	$\left\lvert \sqrt{p_\beta} \right\rangle = \frac{e^{-\beta h^{\mathrm{cl}}/2}}{\sqrt{Z(\beta)}} \lvert \mathbf{1} \rangle$ $H \left\lvert \sqrt{p_\beta} \right\rangle = 0$
Detailed balance condition	$L_i^T = e^{\beta h^{\mathrm{cl}}} L_i e^{-\beta h^{\mathrm{cl}}}$	$H_i^\dagger = H_i^T = H_i$
Spectrum	$e^{Lt} \lvert p_i \rangle = e^{\lambda_i t} \left\lvert p_{(i)} \right\rangle \to \lvert p_\beta \rangle$	$\lambda_i = \left\langle \Phi_{(i)} \right\rvert H \left\lvert \Phi_{(i)} \right\rangle$

Spectral gap optimization

- Goal: optimize $\tilde{\gamma}_{in}$ to enhance the spectral gap $-\lambda_1$

- With first excited state $|\Phi_1\rangle$, gap derivatives with respect to kinetic coefficients $\tilde{\gamma}$ follow from **Hellmann–Feynman theorem**

$$-\frac{\partial \lambda_1}{\partial \tilde{\gamma}_{in}} = \langle \Phi_1 | \frac{\partial H}{\partial \tilde{\gamma}_{in}} | \Phi_1 \rangle$$

- The gap is extensive with $\tilde{\gamma}$. Define the normalized gap as

$$\Delta = \frac{\lambda_1}{\mathrm{tr}(L)} \dim(L)$$

Symmetry

G_Φ: symmetry group of $|\Phi_0\rangle\langle\Phi_0|$
G_H: symmetry group of H

Maximize the gap Δ ➡ $G_H = G_\Phi$

- **Concavity:** $G_H \geq G_\Phi$

Reduce many free parameters in $\tilde{\gamma}_{in}$

The gap increases when H obeys the symmetry of $|\Phi_0\rangle$.

- **Spontaneous symmetry breaking:** $G_H \leq G_\Phi$

Otherwise, $U_{g_H}|\Phi_0\rangle$ is another ground state.

Kshiti Sneh Rai, Jin-Fu Chen, Patrick Emonts, Jordi Tura, Spectral Gap Optimization for Enhanced Adiabatic State Preparation, arXiv:2409.15433, accepted in PRL

Application: Kinetic Curie-Weiss model

- Classical Hamiltonian

$$h^{\mathrm{cl}} = -\frac{1}{N} J \sum_{i<j} \sigma_i^z \sigma_j^z.$$

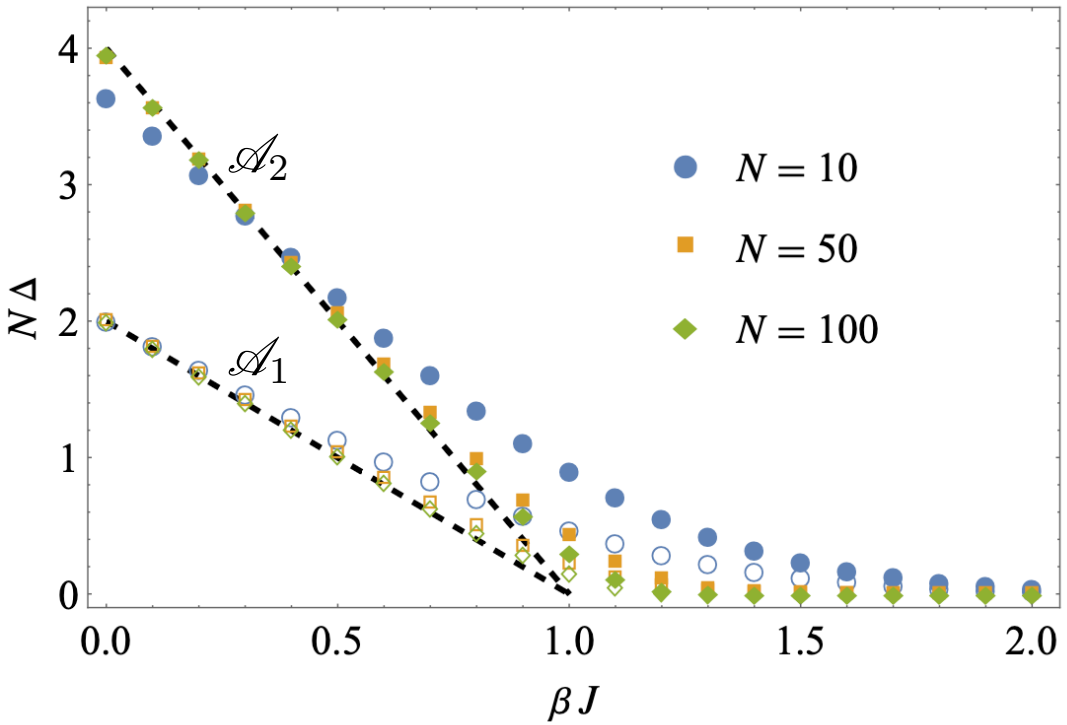
- Single spin flip:

$$\tilde{\mathscr{A}_1} = \{\sigma_i^x | 1 \le i \le N\}$$

- Double spin flip:

$$\tilde{\mathscr{A}_2} = \{\sigma_i^x, \sigma_i^x \sigma_j^x | 1 \le i < j \le N\}$$

Permutation symmetry
Local Z2 symmetry



2-nd phase transition at $\beta J = 1$
Spectral gap closes for large N (ergodicity breaking)

Application: 1D Kinetic Ising model model

- Commuting local terms

$$h^{\mathrm{cl}} = -\sum_{i=1}^{N} J\sigma_i^z\sigma_{i+1}^z.$$

- Single spin flip

$$\tilde{\mathscr{A}}_1 = \{\sigma_i^x | 1 \le i \le N\}$$

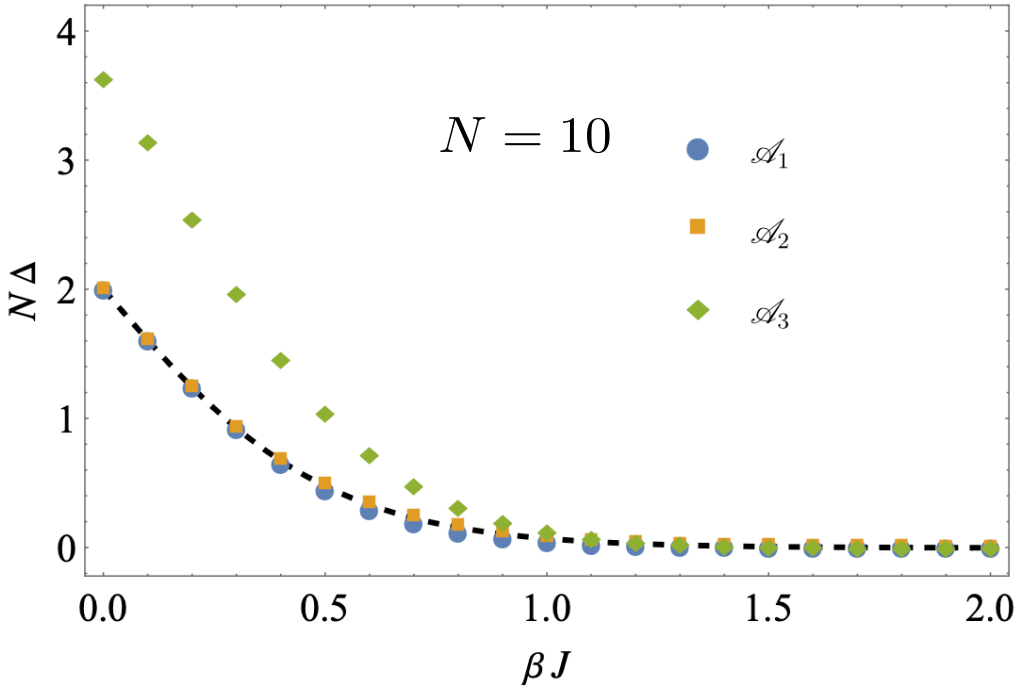
$$\tilde{\mathscr{A}}_2 = \{\tilde{A}_{i1}, \tilde{A}_{i2} | 1 \le i \le N\}$$

$$\tilde{A}_{i12} = (1 \pm \sigma_{i-1}^z\sigma_{i+1}^z)\sigma_i^x/2$$

- Double spin flip

$$\tilde{\mathscr{A}}_3 = \{\sigma_i^x, \sigma_i^x\sigma_{i+1}^x | 1 \le i \le N\}$$

Translational symmetry
Local Z2 symmetry



Dashed curve: Glauber dynamics

$$N\Delta = \frac{2[1-\tanh(\beta J)]^2}{1+\tanh(\beta J)^2}$$

arXiv:2411.03420

Preparing quantum thermal states

- **Preparing quantum thermal states is challenging**
- It is **QMA-hard to prepare the ground state of a local Hamiltonian**—also for low-temperature thermal states.
- To what extent can current methods **prepare quantum thermal states?**
- Classical Liouvillians (diagonal) ➡ Lindbladians (density matrix)

$$\mathcal{L}[\bullet] = -i[K, \bullet] + \sum_n \gamma_n (A_n \bullet A_n^\dagger - \frac{1}{2}\{A_n^\dagger A_n, \bullet\})$$

unitary jump dissipation

- Lindblad operator set: $\mathscr{A} = \{A_n\}$

Lindbladian

- The adjoint Lindblad equation

$$\mathcal{L}^\sharp[\bullet] = i[K, \bullet] + \sum_n \gamma_n (A_n^\dagger \bullet A_n - \frac{1}{2}\{A_n^\dagger A_n, \bullet\})$$

- Quantum detailed balance condition:

$$\exists \text{ Hermitian } N, \text{ such that}$$

$$\mathcal{L}^\sharp[\bullet] - \rho_\beta^{-1/2}\mathcal{L}[\rho_\beta^{1/2} \bullet \rho_\beta^{1/2}]\rho_\beta^{-1/2} = 2i[N, \bullet]$$

$$[N, \rho_\beta] = 0, \qquad \rho_\beta = e^{-\beta h}/Z(\beta)$$

- Decompose it into $\mathcal{L}[\bullet] = \mathcal{N}[\bullet] + \mathcal{D}[\bullet]$

$$\mathcal{N}[\bullet] = -i[N, \bullet] = -\mathcal{N}^\sharp[\bullet],$$

$$\mathcal{D}^\sharp[\bullet] = \rho_\beta^{-1/2}\mathcal{D}[\rho_\beta^{1/2} \bullet \rho_\beta^{1/2}]\rho_\beta^{-1/2}$$

F. Fagnola, V. Umanità, *Commun. Math. Phys.* **298**, 523–547 (2010).

Quantum detailed balance condition

- Infinite-temperature Lindbladian

$$\tilde{\mathcal{D}}[\bullet] = \sum_n \tilde{\gamma}_n (\tilde{A}_n \bullet \tilde{A}_n^\dagger - \frac{1}{2}\{\tilde{A}_n^\dagger \tilde{A}_n, \bullet\})$$

- Detailed balance condition

$$\tilde{\mathcal{D}}^\sharp[\bullet] = \tilde{\mathcal{D}}[\bullet] \quad \Longrightarrow \quad \sum_n \tilde{\gamma}_n \tilde{A}_n \bullet \tilde{A}_n^\dagger = \sum_n \tilde{\gamma}_n \tilde{A}_n^\dagger \bullet \tilde{A}_n$$

Pairs of Lindblad operators:

$$\tilde{A}_{n'} = \tilde{A}_n^\dagger \in \tilde{\mathscr{A}}$$

$$\tilde{\gamma}_{n'} = \tilde{\gamma}_n$$

arXiv:2411.03420

Finite-temperature Lindbladian

- To prepare $\rho_\beta \propto e^{-\beta H}$, finite-temperature Lindbladian:

$$\mathcal{D}[\bullet] = -i[K, \bullet] + \sum_n \tilde{\gamma}_n (A_n \bullet A_n^\dagger - \frac{1}{2}\{A_n^\dagger A_n, \bullet\}).$$

unitary jump dissipation

- A_n and K are solved from

$$A_n = \rho_\beta^{1/4} \tilde{A}_n \rho_\beta^{-1/4}$$

$$\rho_\beta^{-\frac{1}{4}} K \rho_\beta^{\frac{1}{4}} + \rho_\beta^{\frac{1}{4}} K \rho_\beta^{-\frac{1}{4}} = \frac{i}{2}[\rho_\beta^{-\frac{1}{2}}, \sum_n \tilde{\gamma}_n \tilde{A}_n^\dagger \rho_\beta^{\frac{1}{2}} \tilde{A}_n].$$

arXiv:2411.03420

Parent Hamiltonian

- Parent Hamiltonian in thermofield double space

$$\mathcal{H}[\cdot] \equiv -\rho_\beta^{-1/4}\mathcal{D}[\rho_\beta^{1/4}\cdot\rho_\beta^{1/4}]\rho_\beta^{-1/4}.$$

$$\Downarrow \quad \mathcal{D}^\sharp[\cdot] = \rho_\beta^{-1/2}\mathcal{D}[\rho_\beta^{1/2}\cdot\rho_\beta^{1/2}]\rho_\beta^{-1/2}$$

$$\mathcal{H}[\bullet] = \mathcal{H}^\sharp[\bullet]$$

- It can be constructed as

$$\mathcal{H}[\bullet] = \frac{1}{2}\{O,\bullet\} - \sum_n \tilde{\gamma}_n \tilde{A}_n \bullet \tilde{A}_n^\dagger,$$

Symmetric logarithmic derivative

$$\frac{1}{2}\{O,\rho_\beta^{1/2}\} = \sum_n \tilde{\gamma}_n \tilde{A}_n^\dagger \rho_\beta^{1/2} \tilde{A}_n.$$

arXiv:2411.03420

Quantum vs. Classical

Quantum Lindbladian under quantum detailed balance condition

$$\mathcal{D}^{\sharp}[\bullet] = \rho_\beta^{-1/2}\mathcal{D}[\rho_\beta^{1/2} \bullet \rho_\beta^{1/2}]\rho_\beta^{-1/2}$$

Parent Hamiltonian in extended Hilbert space

$$\mathcal{H}[\cdot] \equiv -\rho_\beta^{-1/4}\mathcal{D}[\rho_\beta^{1/4} \cdot \rho_\beta^{1/4}]\rho_\beta^{-1/4}$$

e.g., Lipkin-Meshkov-Glick model

Classical Liouvillian under detailed balanced condition

$$L^T = e^{\beta h^{\mathrm{cl}}} L e^{-\beta h^{\mathrm{cl}}}$$

Parent Hamiltonian

$$H = -\rho_\beta^{-1/2} L \rho_\beta^{1/2}$$

e.g., Kinetic Ising models

Lipkin-Meshkov-Glick model

$$\tilde{\beta}J = N\beta J = 5$$

$$\tilde{\Delta} = \Delta N(N+2)/4$$

- **LMG Hamiltonian**

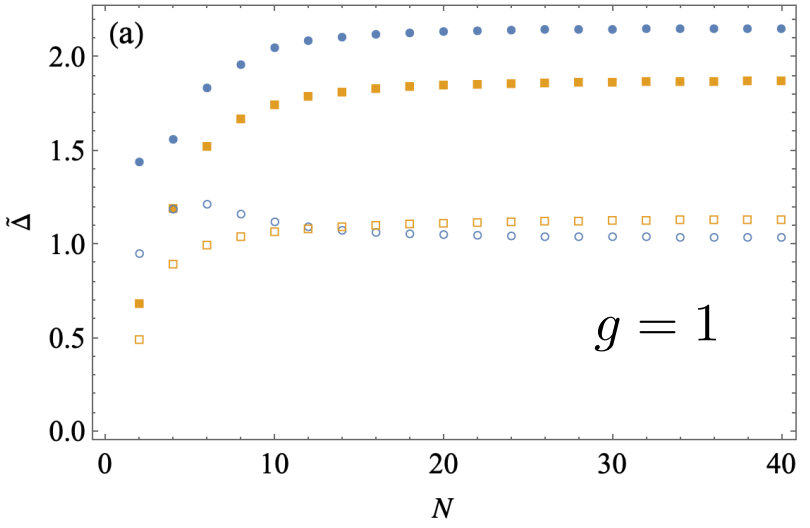
$$h = -J\left(\frac{1}{N}\sum_{i<j}\sigma_i^z\sigma_j^z + r\frac{1}{N}\sum_{i<j}\sigma_i^y\sigma_j^y + g\sum_i \sigma_i^x\right)$$

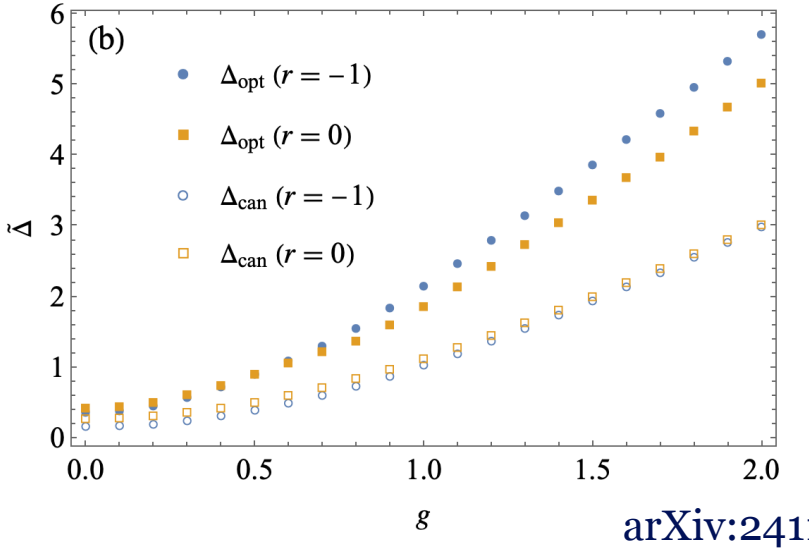
- **In the symmetric subspace,**

$$h = -J\left(\frac{1}{N}2S_z^2 + r\frac{1}{N}2S_y^2 + 2gS_x - \frac{1+r}{2}\right)$$

$$S_k = \sum_{i=1}^{N}\sigma_i^k/2$$

- Construct $\mathcal{L}$ with $A_k = \rho_\beta^{1/4} S_k \rho_\beta^{-1/4}$ and K
- We optimize $\tilde{\gamma}_x, \tilde{\gamma}_y$ and $\tilde{\gamma}_z$ to maximize Δ





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Application: 1D Transverse Ising model

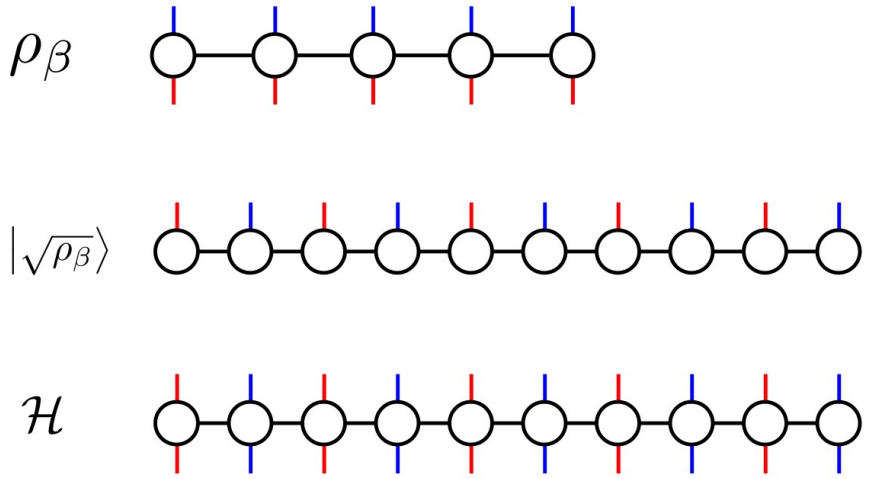
- Transverse Ising Hamiltonian

$$h = -J \sum_{i=1}^{N} \left(\sigma_i^z \sigma_{i+1}^z + g \sigma_i^x \right)$$

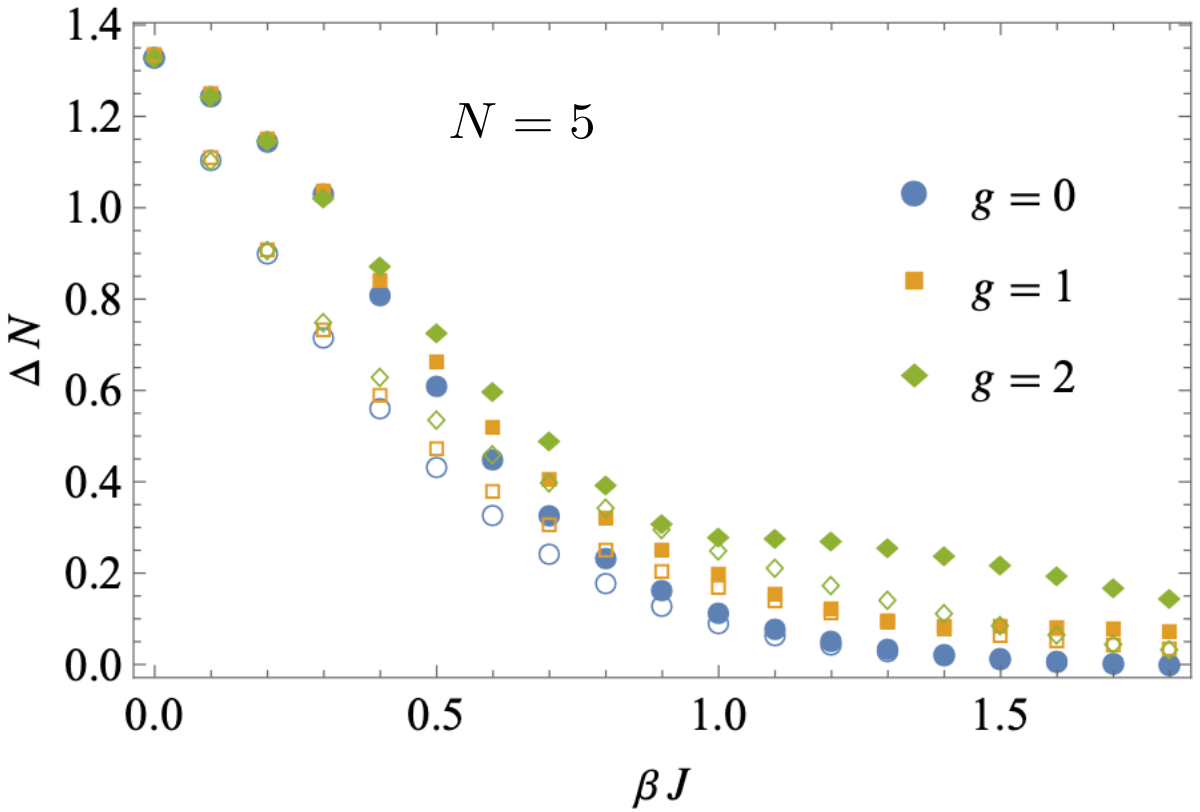
- Infinite-temperature Lindblad operators

$$\tilde{\mathscr{A}} = \{\sigma_i^x, \sigma_i^y, \sigma_i^z\}$$

- MPS representation:



Translational symmetry
Local Z2 symmetry



Filled: optimal kinetic coefficients
Empty: canonical kinetic coefficients

Summary

Thermodynamics

- A family of Lindbladians under quantum detailed balance.
- Relevant to quantum thermodynamics, simulation, and algorithms.

Gap optimization

- Engineering transient dynamics for fast thermalization.
- Symmetry reduces free parameters in $\tilde{\gamma}$
- Maximizing Δ.

Future scope

- Study stochastic thermodynamics for these Lindbladians.
- Optimizing control of open quantum many-body systems.
- MPS Variational and SDP bounds for larger systems.



Thank You

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