Optimizing Brownian heat engine with shortcut strategy

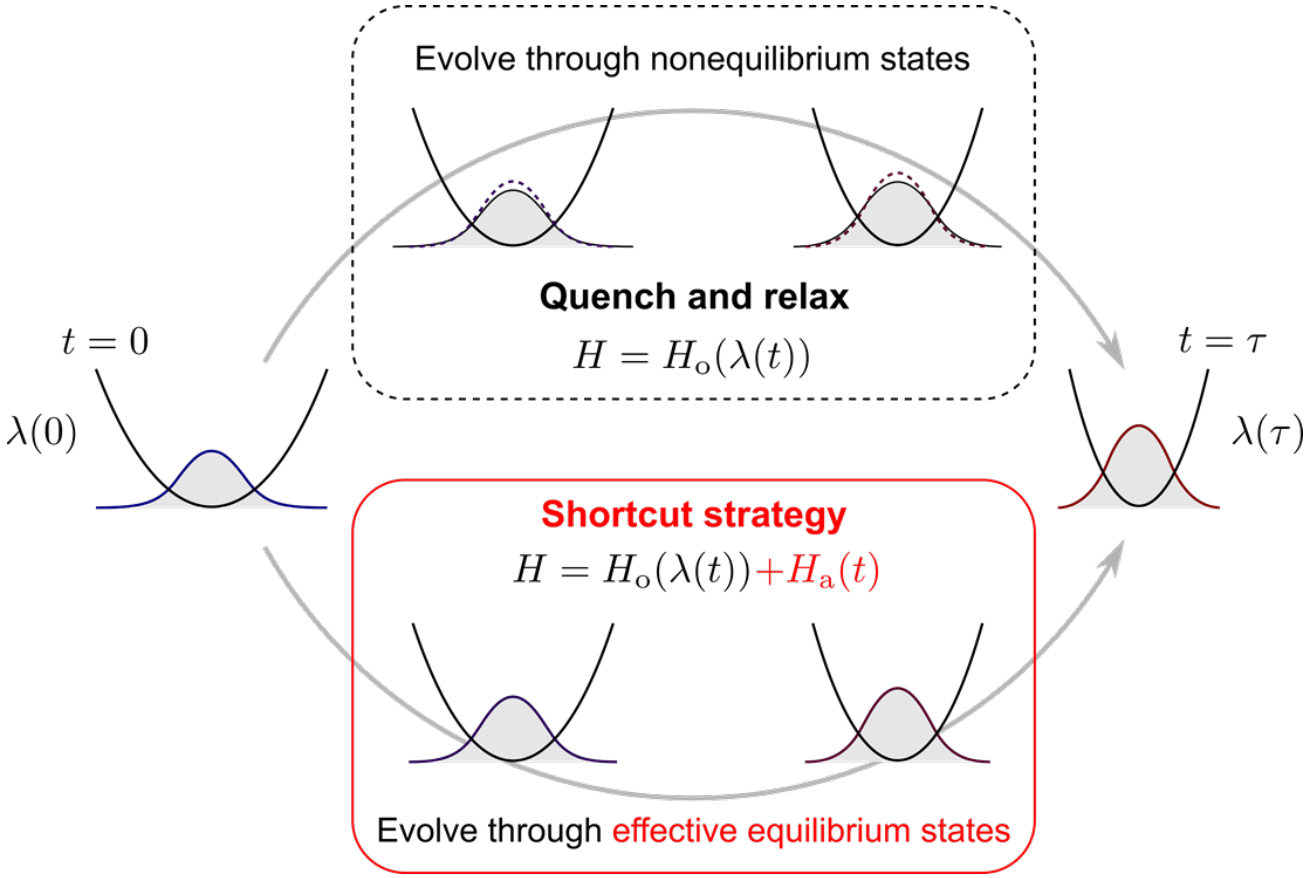
XXVII Sitges Conference on Statistical Mechanics

Jinfu Chen

Peking University

31 May, 2023

Transformation between two equilibrium states



I. A. Martínez, A. Petrosyan, D. Guéry-Odelin, E. Trizac, and S. Ciliberto, Nat. Phys. 12, 843 (2016).
G. Li, H. T. Quan and Z. C. Tu, Phys. Rev. E 96, 012144 (2017).

Outline

- Shortcut strategy for underdamped Brownian motion
- Optimizing Brownian heat engine with shortcut strategy
- Discussions on underdamped and overdamped limits
- Summary and outlook

Shortcut strategy of evolution

- Consider a Brownian particle trapped in a potential.
- Goal: let the system evolve along

$$\rho(t)=\rho_{\mathrm{ieq}}(t)=\frac{1}{Z(\beta(t))}e^{-\beta(t)H_{\mathrm{o}}(\lambda(t))}$$

- Total Hamiltonian: $H=H_{\mathrm{o}}(\lambda(t))+H_{\mathrm{a}}(t)$

origin auxiliary

- Dynamics: the Fokker-Planck equation

$$\frac{\partial\rho}{\partial t}=\mathscr{L}\left[\rho\right]+\mathscr{D}\left[\rho\right],$$

Deterministic $$\mathscr{L}\left[\rho\right]=-\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial p}\rho\right)+\frac{\partial}{\partial p}\left(\frac{\partial H}{\partial x}\rho\right),$$

Dissipative $$\mathscr{D}\left[\rho\right]=\kappa m\frac{\partial}{\partial p}\left(\frac{\partial H}{\partial p}\rho+\frac{1}{\beta}\frac{\partial\rho}{\partial p}\right).$$

Shortcuts to isothermality: G. Li, H. T. Quan and Z. C. Tu, Phys. Rev. E 96, 012144 (2017)

Shortcut strategy of evolution

- We consider harmonic potential

$$H_{\mathrm{o}} = \frac{1}{2m}p^2 + U_{\mathrm{o}}, \qquad U_{\mathrm{o}} = \frac{1}{2}m\lambda^2 x^2,$$

- The auxiliary Hamiltonian is solved in the form

$$H_{\mathrm{a}}(x,p,t) = \dot{\lambda}h_\lambda(x,p,\lambda) + \dot{\beta}h_\beta(x,p,\lambda),$$

- where

$$h_\lambda = \frac{1}{2\kappa}\left[\frac{1}{\lambda m}(p-\kappa m x)^2 + m\lambda x^2\right],$$
$$h_\beta = \frac{p^2 + (p-\kappa m x)^2 + 2m^2\lambda^2 x^2}{4\kappa\beta m}.$$

Y. Jun and P. Y. Lai, Phys. Rev. Res. 3, 033130 (2021)

Energetics in shortcut processes

- The average input power and the average heat flux are

$$\dot{W} \equiv \left\langle \frac{\partial H}{\partial t} \right\rangle = \dot{W}_{\rm o} + \dot{W}_{\rm irr}, \qquad \dot{Q} \equiv \iint H \frac{\partial \rho_{\rm ieq}}{\partial t} dx dp = \dot{Q}_{\rm o} + \dot{Q}_{\rm irr},$$

- We divide work and heat into quasistatic and irreversible parts

$$\dot{W}_{\rm o} = \dot{\lambda} \left\langle \frac{\partial H_{\rm o}}{\partial \lambda} \right\rangle, \qquad \dot{Q}_{\rm o} = \iint H_{\rm o} \frac{\partial \rho_{\rm ieq}}{\partial t} dx dp,$$

$$\dot{W}_{\rm irr} = \left\langle \frac{\partial H_{\rm a}}{\partial t} \right\rangle, \qquad \dot{Q}_{\rm irr} = \iint H_{\rm a} \frac{\partial \rho_{\rm ieq}}{\partial t} dx dp.$$

- First law:

$$\frac{\partial \langle H_{\rm o} \rangle}{\partial t} = \dot{W}_{\rm o} + \dot{Q}_{\rm o}, \qquad \frac{\partial \langle H_{\rm a} \rangle}{\partial t} = \dot{W}_{\rm irr} + \dot{Q}_{\rm irr}.$$

$$W_{\rm irr} = \underset{\|}{\langle H_{\rm a} \rangle (\tau^+)} - \underset{\|}{\langle H_{\rm a} \rangle (0^-)} - Q_{\rm irr}$$

$H_{\rm a}$ is switched off initially and finally $\quad 0 \qquad\qquad 0$

- Maximizing the output work $-W_{\rm o} - W_{\rm irr}$ = maximizing $Q_{\rm irr}$

Thermodynamic length of irreversibility

- For shortcut processes, the irreversible heat is exactly

$$Q_{\rm irr} = -\int_0^\tau \begin{pmatrix} \dot\lambda & \dot\beta \end{pmatrix} \mathbf{g} \begin{pmatrix} \dot\lambda \\ \dot\beta \end{pmatrix} dt,$$

Thermodynamic length:

$$\mathcal{L} \equiv \int_0^1 \sqrt{\begin{pmatrix} \tilde\lambda'(s) & \tilde\beta'(s) \end{pmatrix} \mathbf{g} \begin{pmatrix} \tilde\lambda'(s) \\ \tilde\beta'(s) \end{pmatrix}} ds.$$

metric:

$$\mathbf{g} = \kappa m \begin{pmatrix} \left\langle \frac{\partial h_\lambda}{\partial p} \frac{\partial h_\lambda}{\partial p} \right\rangle & \left\langle \frac{\partial h_\lambda}{\partial p} \frac{\partial h_\beta}{\partial p} \right\rangle \\ \left\langle \frac{\partial h_\beta}{\partial p} \frac{\partial h_\lambda}{\partial p} \right\rangle & \left\langle \frac{\partial h_\beta}{\partial p} \frac{\partial h_\beta}{\partial p} \right\rangle \end{pmatrix}$$

- The irreversible work (or heat) is bounded by

$$W_{\rm irr} = -Q_{\rm irr} \geq \frac{\mathcal{L}^2}{\tau} > 0$$

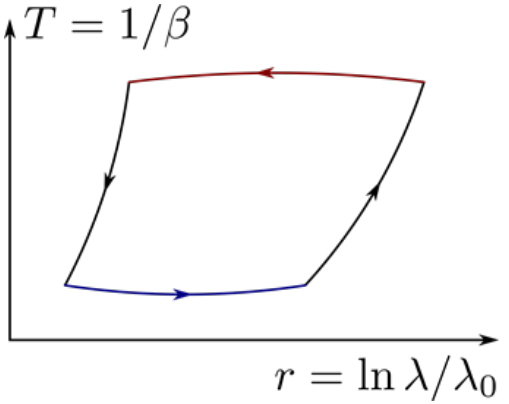
- Equality is reached for the optimal protocol.

- We next consider cyclic control of a Brownian heat engine.

G. Li, JFC, C. P. Sun, H. Dong, Phys. Rev. Lett. 128, 230603 (2022)
JFC, Phys. Rev. E 106, 054108 (2022)

Brownian heat engine: maximum power

- Consider a closed path of a heat-engine cycle $\lambda(s)$ and $\beta(s)$ in control parameter space.



- The output work is bounded by

$$-W = -W_{\rm o} - W_{\rm irr} \leq -W_{\rm o} - \mathcal{L}^2/\tau,$$

$W_{\rm o}$: work of quasistatic cycle $\mathcal{L}$: thermodynamic length of closed path

- Choose the operation time $\tau_{\rm max} = 2\mathcal{L}^2/(-W_{\rm o})$.

- The maximum power is

.

$$P_{\rm max} = \frac{-W}{\tau_{\rm max}} = \frac{(-W_{\rm o})^2}{4\mathcal{L}^2}.$$

JFC, Phys. Rev. E 106, 054108 (2022)

Shortcut Brownian heat engine: EMP

- Divide the path into heat absorbed and heat released paths according to $\dot{Q} > 0$ and $\dot{Q} < 0$, and define their thermodynamic lengths $\mathcal{L}_+$ and $\mathcal{L}_-$.
- Heat absorbed in a cycle $Q_+ = Q_{\mathrm{o},+} - \mathcal{L}_+\mathcal{L}/\tau.$
- Efficiency of the cycle:

$$\eta \equiv \frac{-W}{Q_+} = \frac{-W_\mathrm{o} - \mathcal{L}^2/\tau}{-W_\mathrm{o}/\eta_\mathrm{o} - \mathcal{L}_+\mathcal{L}/\tau}.$$

where the efficiency of quasistatic cycle is

$$\eta_\mathrm{o} = \frac{-W_\mathrm{o}}{Q_{\mathrm{o},+}}.$$

- Efficiency at maximum power:

$$\eta_\mathrm{EMP} = \frac{\eta_\mathrm{o}}{2 - \eta_\mathrm{o}\mathcal{L}_+/\mathcal{L}}.$$

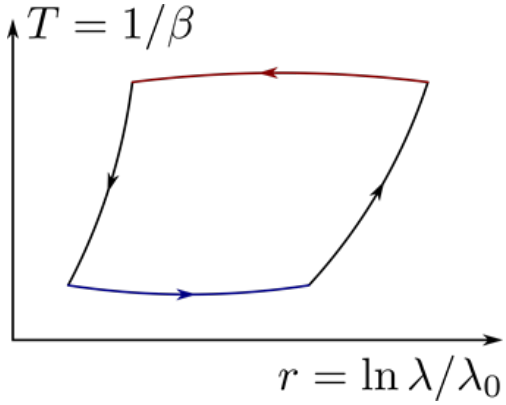
Results for harmonic potential

- Represent the control parameters as

$$\lambda = \lambda_0 e^r, \quad \beta = 1/T.$$

- Quasistatic output work of the cycle:

$$-W_{\mathrm{o}} = -\oint T dr.$$



- Proportional to the area $\mathcal{A} = \left|\oint T dr\right|$ in $r - T$ diagram.

- Thermodynamic length of the cycle

$$\mathcal{L} = \oint \sqrt{\frac{(Tdr - dT)^2}{\kappa T} + \frac{\kappa}{(\lambda_0 e^r)^2}\frac{1}{T}\left(Tdr - \frac{dT}{2}\right)^2}.$$

Dissipation in underdamped limit

$\kappa \ll \lambda$

Dissipation in overdamped limit

$\kappa \gg \lambda$

Optimization of heat-engine cycle

- For the optimal cycle on a given closed path, the power is

$$P = \frac{\mathcal{A}}{\tau} - \frac{\mathcal{L}^2}{\tau^2},$$

where $\mathcal{A}$ and $\mathcal{L}$ are the area and the thermodynamic length of the closed path.

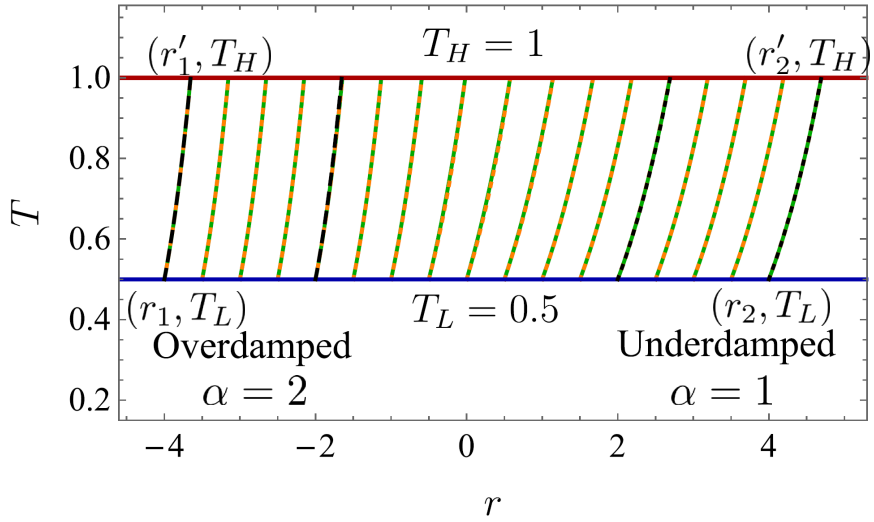
- Two aspects for optimizing cycles with shortcut strategy:
  1. Along a given closed path, manipulate the control parameters with a proper constant velocity to reach the maximum power

$$P_{\max} = \frac{\mathcal{A}^2}{4\mathcal{L}^2}.$$

  2. Find out a closed path with large $\mathcal{A}/\mathcal{L}$.

Optimization in general damped situation

- Choose two isothermal lines (T_L and T_H) and two connecting paths to form a closed cycle.



- Connecting paths are shortest geodesic curves or exponential paths

$$T(r) = T_L e^{\alpha(r-r_0)}, \qquad r_0 \le r \le r_0 + \frac{1}{\alpha}\ln(\frac{T_H}{T_L}).$$

- We choose $\alpha_{1,2}$ to minimize $\mathcal{L}_{\exp}(r_1, \alpha_1, r_2, \alpha_2)$ or fixed as $\alpha_{1,2} = 1$ or 2.

Underdamped limit

- In the underdamped limit $\kappa \ll \lambda$, the thermodynamic length is similified to

$$\mathcal{L} = \oint \sqrt{\frac{(Tdr - dT)^2}{\kappa T}}.$$

- Choose the exponential paths with $\alpha = 1$ (zero length), and obtain

$$P_{\max}^{\text{under}} = \frac{\kappa}{4}(\sqrt{T_H} - \sqrt{T_L})^2,$$

$$\eta_{\text{EMP}}^{\text{under}} = \eta_{\text{CA}} \equiv 1 - \sqrt{\frac{T_L}{T_H}}.$$

- Agree with Curzon and Ahlborn's prediction [Am. J. Phys 43, 22 (1975)].

Overdamped limit

- In the overdamped limit $\kappa \gg \lambda$, $\mathcal{L} = \oint \sqrt{\frac{\kappa}{(\lambda_0 e^r)^2}\frac{1}{T}\left(Tdr - \frac{dT}{2}\right)^2}$.
- Choose the exponential paths with $\alpha = 2$ (zero length), and obtain

$$P_{\max}^{\text{over}} = \frac{\lambda_0^2 (T_H - T_L)^2 (r_2 - r_1)^2}{16\kappa T_L (e^{-r_1} - e^{-r_2})^2},$$

$$\eta_{\text{EMP}}^{\text{over}} = \frac{\eta_{\text{C}}}{2 + \left[\frac{1}{(r_2 - r_1)} - \frac{1}{2}\right]\eta_{\text{C}}}.$$

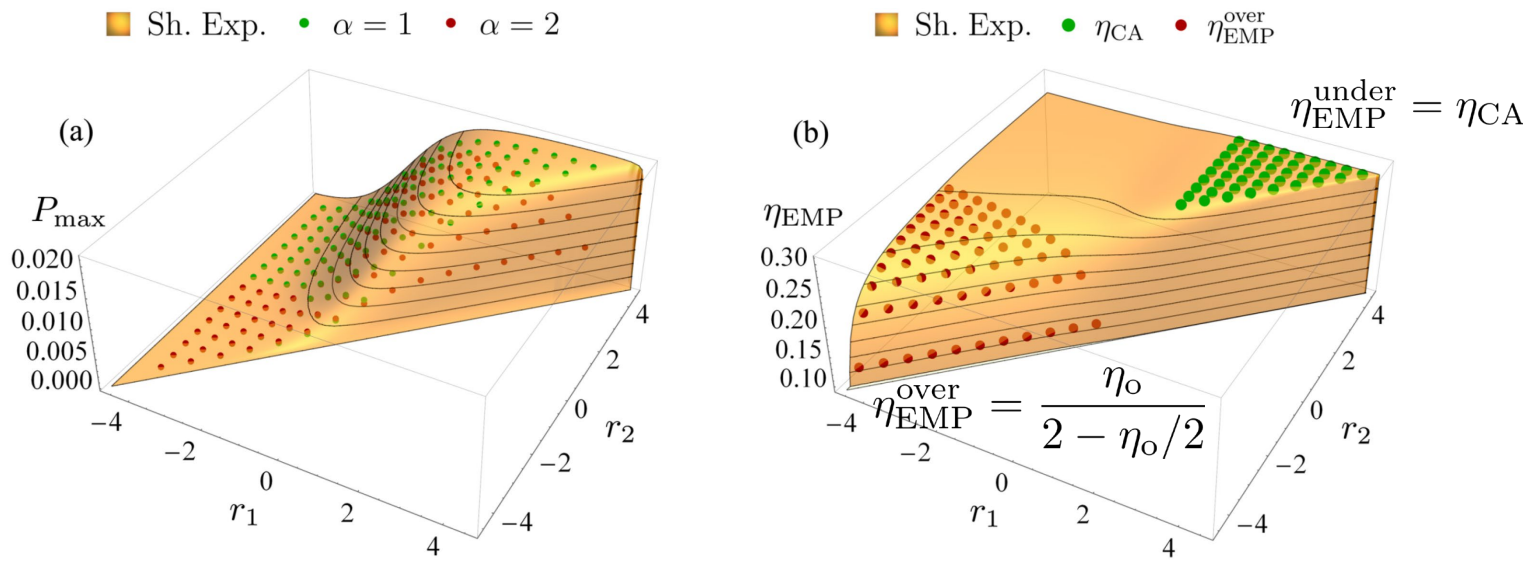
- The quasistatic efficiency is

$$\eta_{\text{o}}^{\text{over}} = \frac{\eta_{\text{C}}}{1 + \eta_{\text{C}}/[2(r_2 - r_1)]} < \eta_{\text{C}}.$$

Heat leakage from momentum degree of freedom

- EMP matches that in [T. Schmiedl and U. Seifert, Europhys. Lett. 81, 20003 (2007)].

$P_{\max}$ and η_{EMP} in general damped situation

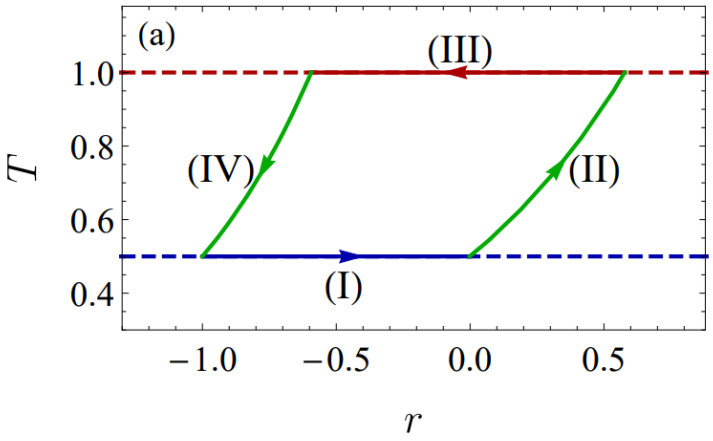


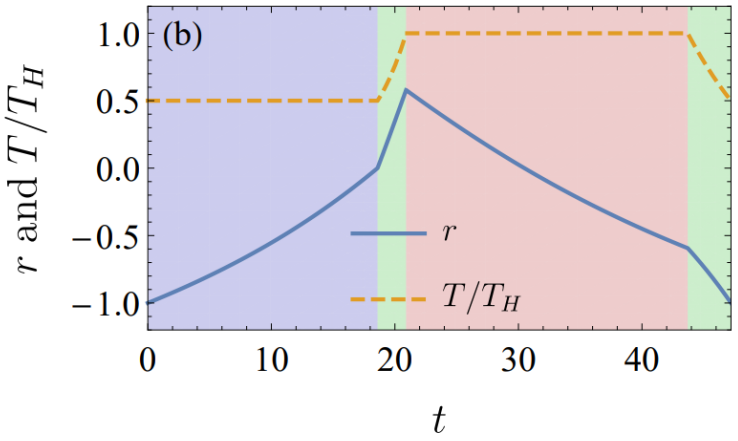
- $P_{\max}$ is the largest with the shortest exponential paths.
- Underdamped regime ($r \gg 0$): large $P_{\max}$, Curzon-Ahlborn's type.
- Overdamped regime ($r \ll 0$): small $P_{\max}$, Schmiedl-Seifert's type.

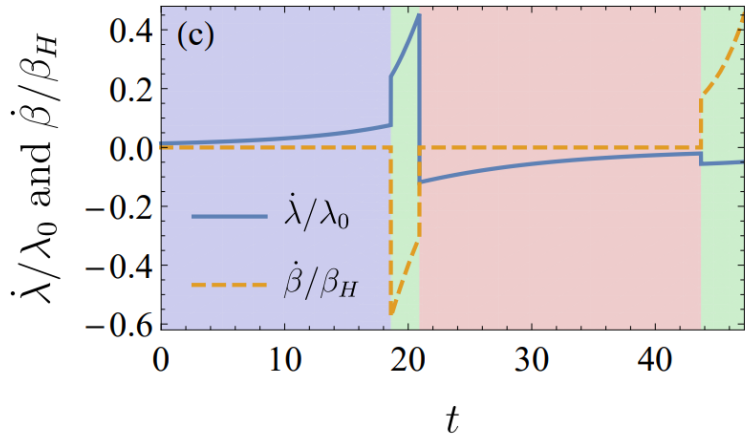
Control of the cycle

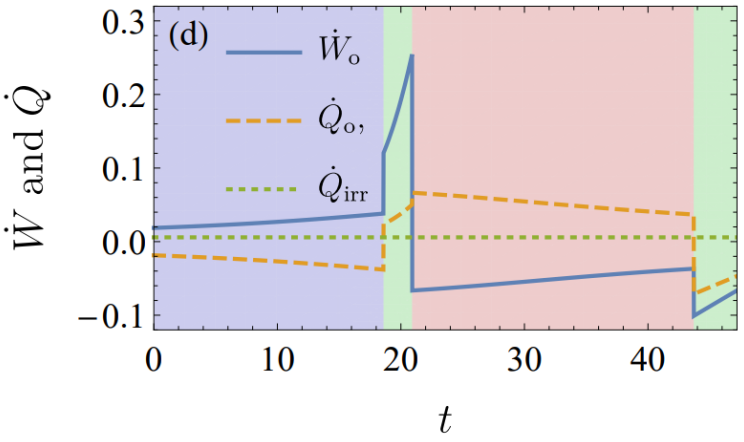
- The control of a typical cycle

Connecting processes consume time according to thermodynamic length



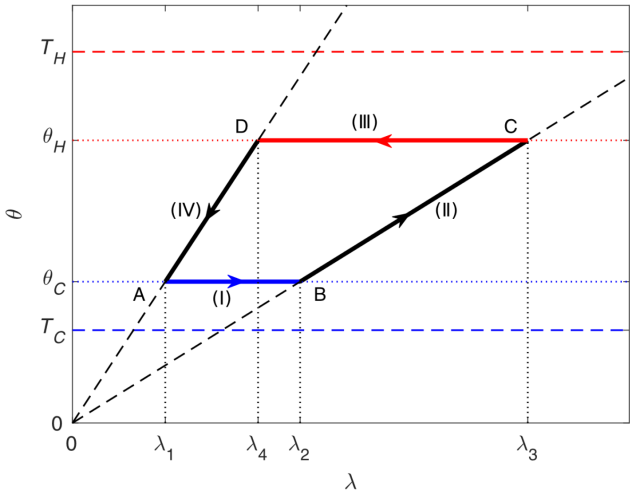


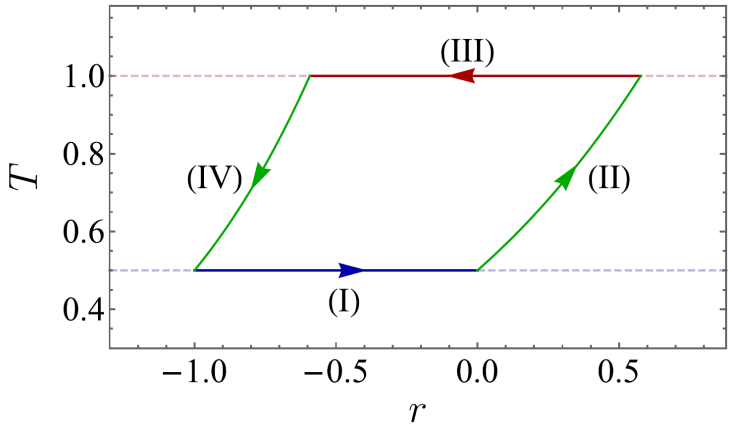




JFC, Phys. Rev. E 106, 054108 (2022)

Traditional optimization vs. Shortcut strategy





- Given protocol, solve cycle diagram.
- Irreversibility: deviation from quasi-static cycles.
- Thermodynamic length valid for slow driving processes.
- Optimal cycles are only solved in overdamped limit [T. Schmiedl and U. Seifert, Europhys. Lett. 81, 20003 (2007)]
- and underdamped limits [A. Dechant, N. Kiesel, and E. Lutz, Europhys. Lett. 119, 50003 (2017) and Y. H. Chen, JFC, Z. Fei, and H. T. Quan, Phys. Rev. E 106, 024105 (2022)].

- Given cycle diagram, solve $H_{\mathrm{a}}(t)$.
- Irreversibility: energy cost to implement $H_{\mathrm{a}}(t)$.
- Exact thermodynamic length for arbitrary finite-time processes.
- Geometric interpretation of the optimization to acquire large output power

$$P_{\max} = \frac{\mathcal{A}^2}{4\mathcal{L}^2}.$$

Summary and Outlook

- Shortcut strategy: a new way to construct and optimize the Brownian heat engines.
- Generalization to power-law potentials $U_{\rm o}(x,\lambda)=\frac{1}{2n}m\lambda^{n+1}x^{2n}$.
- Strict $1/\tau$ scaling of irreversible heat in shortcut processes.
- The performance of the optimal heat-engine cycle with shortcut strategy is characterized by $\mathcal{A}$ and $\mathcal{L}$.
- Better understanding of the optimal cycles in the underdamped and the overdamped limits.
- Extension to open quantum systems under external driving and also active systems.

JFC, Phys. Rev. E 106, 054108 (2022)



Thank You


PEKING UNIVERSITY
1898


SITGES
CONFERENCE
on Statistical Mechanics

- Approximate shortcuts with momentum-independent auxiliary Hamiltonian see

  G. Li, C. P. Sun, H. Dong, Phys. Rev. E, 107, 014103 (2023)

- Lindblad master equation

$$\dot{\rho} = -i[H, \rho] + \sum_k \gamma_k (L_k \rho L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho - \frac{1}{2} \rho L_k^\dagger L_k).$$