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Quantum Otto heat engine on piston model: the effect of interaction and finite time

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Outline

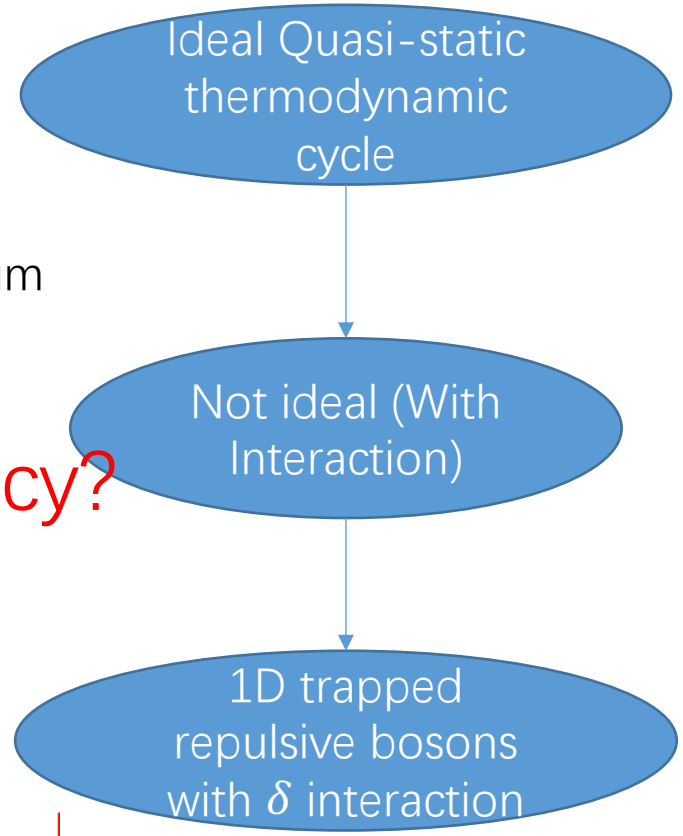
- **Quantum Otto Heat Engine with Trapped Repulsive Bosons**
  - 1D repulsive bosons in a hard wall box
  - The effect of the interaction on the efficiency
- **Finite-time Otto Heat Engine with Piston Model**
  - Finite-time adiabatic process
  - Finite-time Otto heat engine
  - Realization on piston model

Quantum Otto Heat Engine with Trapped Repulsive Bosons

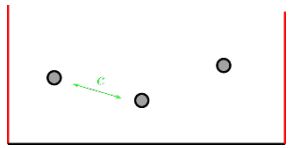
Many-body system

Interaction affects the spectrum

How interaction affects the efficiency?



Bethe ansatz
M. Gaudin, Phys. Rev. A **4**, 386 (1971).

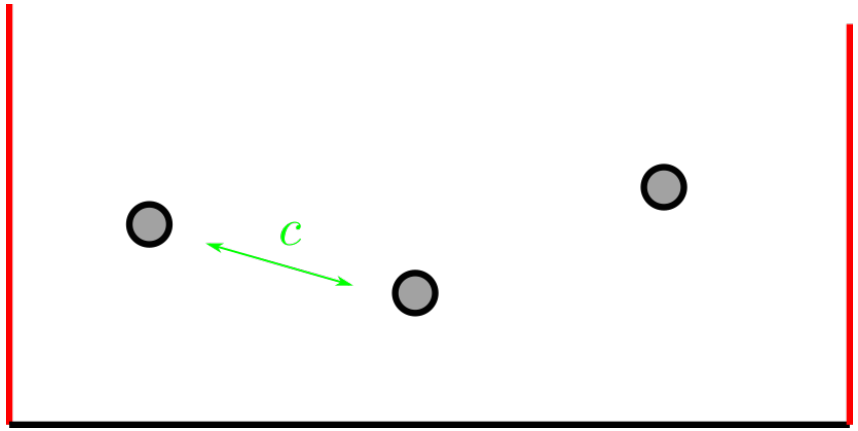


H. T. Quan, Y. X. Liu, C. P. Sun, and F. Nori, Phys. Rev. E **76**, 031105 (2007).

Interacting Bose gas confined in a time-dependent harmonic trap
J. Jaramillo, M. Beau, and A. del Campo, New J. Phys. **18**, 075019 (2016).

Quantum Szilard Engine with Attractively Interacting Bosons
J. Bengtsson, M. N. Tengstrand, A. Wacker, P. Samuelsson, M. Ueda, H. Linke, and S. M. Reimann, Phys. Rev. Lett. **120**, 100601 (2018).

1D Repulsive Bosons in a Hard Wall Box



- The Hamiltonian is

$$H(L,c) = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \frac{c}{m} \sum_{i<j} \delta(x_i - x_j) + V(\{x_j\})$$

1D Repulsive Bosons in a Hard Wall Box

- Bethe's Equation

$$k_i L = \pi n_i + \sum_{j \neq i} (\arctan \frac{c}{k_i - k_j} + \arctan \frac{c}{k_i + k_j}).$$

- The energy

$$E^{(L)}_{\{n_i\}} = \sum_{i=1}^{N} k_i^2 / 2m.$$

- Large c limit $c \to \infty$

$$k_i L = \pi(n_i + i - 1) - \sum_{j \neq i} \left(\frac{k_i - k_j}{c} + \frac{k_i + k_j}{c} \right).$$

- The energy

$$E^{(L)}_{\{n_i\}} = \frac{\pi^2}{2m} \frac{\sum_{i=1}^{N} (n_i + i - 1)^2}{\left(L + \frac{2(N-1)}{c}\right)^2}.$$

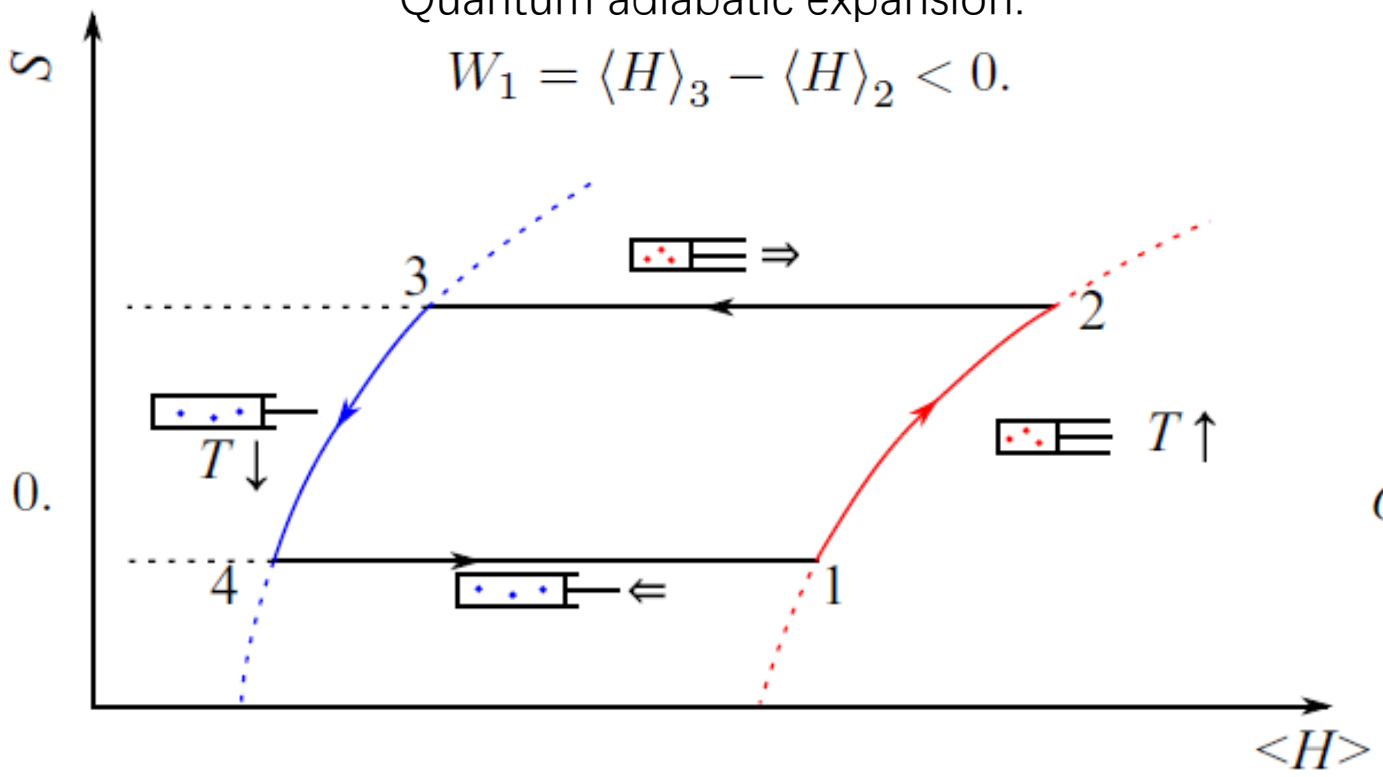
M. Gaudin, Phys. Rev. A **4**, 386 (1971).

Quantum Otto Cycle

For quantum adiabatic processes II & IV, the probability distribution does not change.

Stroke II (2 → 3):
Quantum adiabatic expansion.

$$W_1 = \langle H \rangle_3 - \langle H \rangle_2 < 0.$$



Stroke III (3 → 4):
Isochoric cooling.

$$Q_c = \langle H \rangle_4 - \langle H \rangle_3 < 0.$$

Stroke I (1 → 2):
Isochoric heating.

$$Q_h = \langle H \rangle_2 - \langle H \rangle_1 > 0.$$

Stroke IV (4 → 1):
Quantum adiabatic compressing.

$$W_2 = \langle H \rangle_1 - \langle H \rangle_4 > 0.$$

Interaction Effect on Efficiency

- The efficiency is

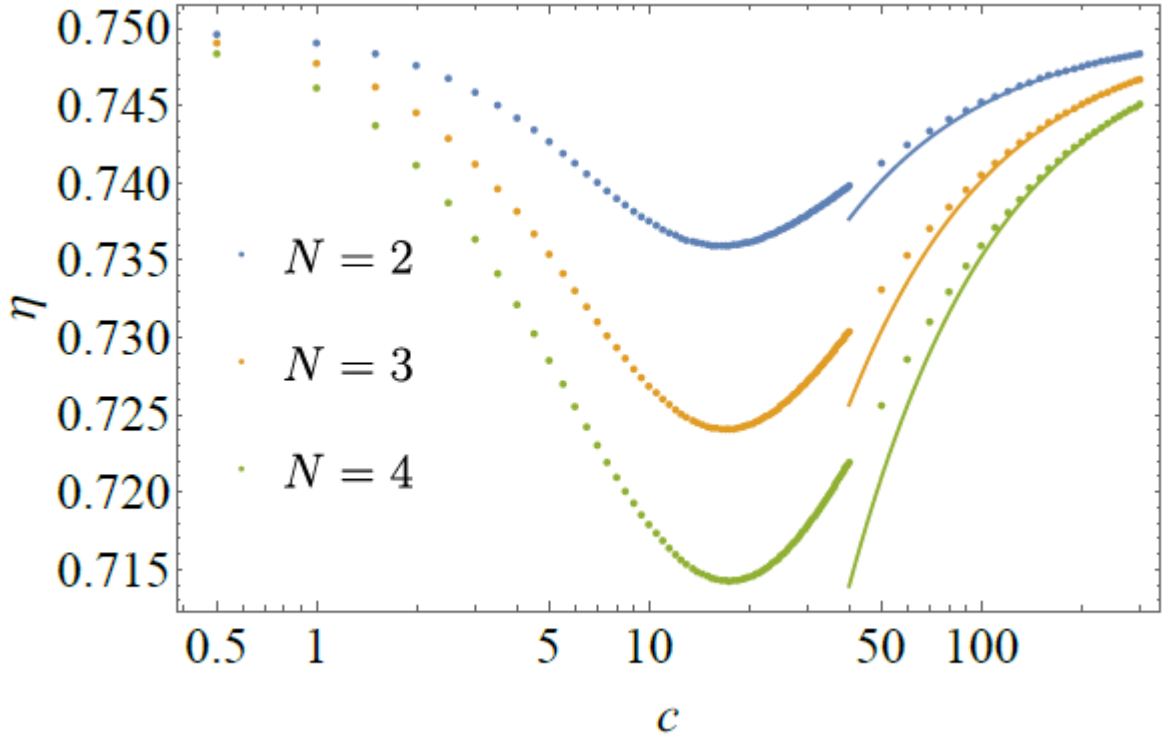
$$\eta = 1 - \frac{|Q_c|}{Q_h} = 1 - \frac{\sum_{\{n_i\}} \left(p^{(2)}_{\{n_i\}} - p^{(4)}_{\{n_i\}} \right) E^{(L_2,c)}_{\{n_i\}}}{\sum_{\{n_i\}} \left(p^{(2)}_{\{n_i\}} - p^{(4)}_{\{n_i\}} \right) E^{(L_1,c)}_{\{n_i\}}}.$$

- For large c, we have

$$\eta = 1 - \left(\frac{L_1 + \frac{2(N-1)}{c}}{L_2 + \frac{2(N-1)}{c}} \right)^2$$

$c \to \infty$

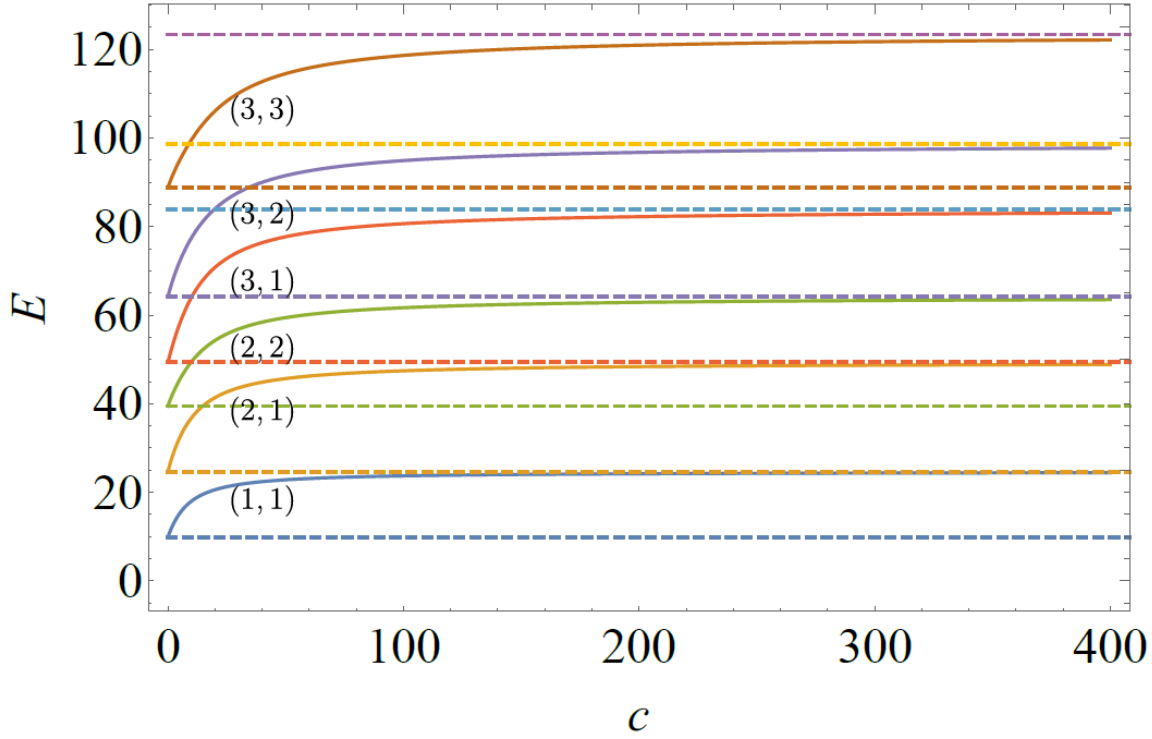
$$\eta_{non} = 1 - \left(\frac{L_1}{L_2} \right)^2$$



Bose-Fermi duality for 1D interacting Bose gas



The Energy for two interacting Bosons



$(n_1, n_2) \to (n_1 + 1, n_2),\ n_1 \geq n_2$

M. Girardeau, J. Math. Phys. 1, 516 (1960).

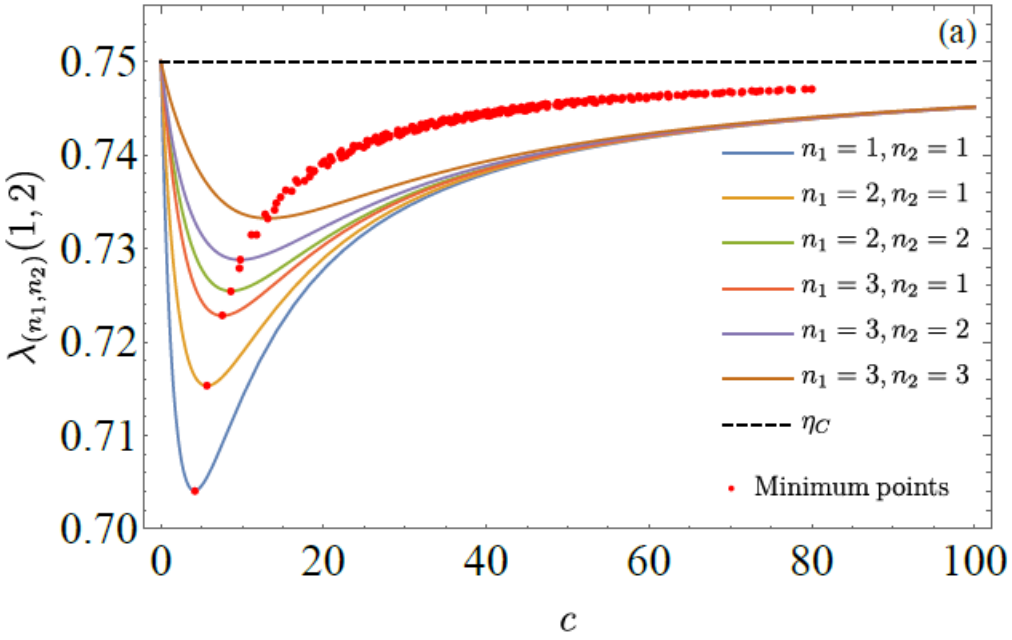
Interaction Effect on Efficiency

- With a ratio between two energy

$$\lambda_{\{n_i\}}(L_1, L_2) = 1 - \frac{E^{(L_2,c)}_{\{n_i\}}}{E^{(L_1,c)}_{\{n_i\}}}$$

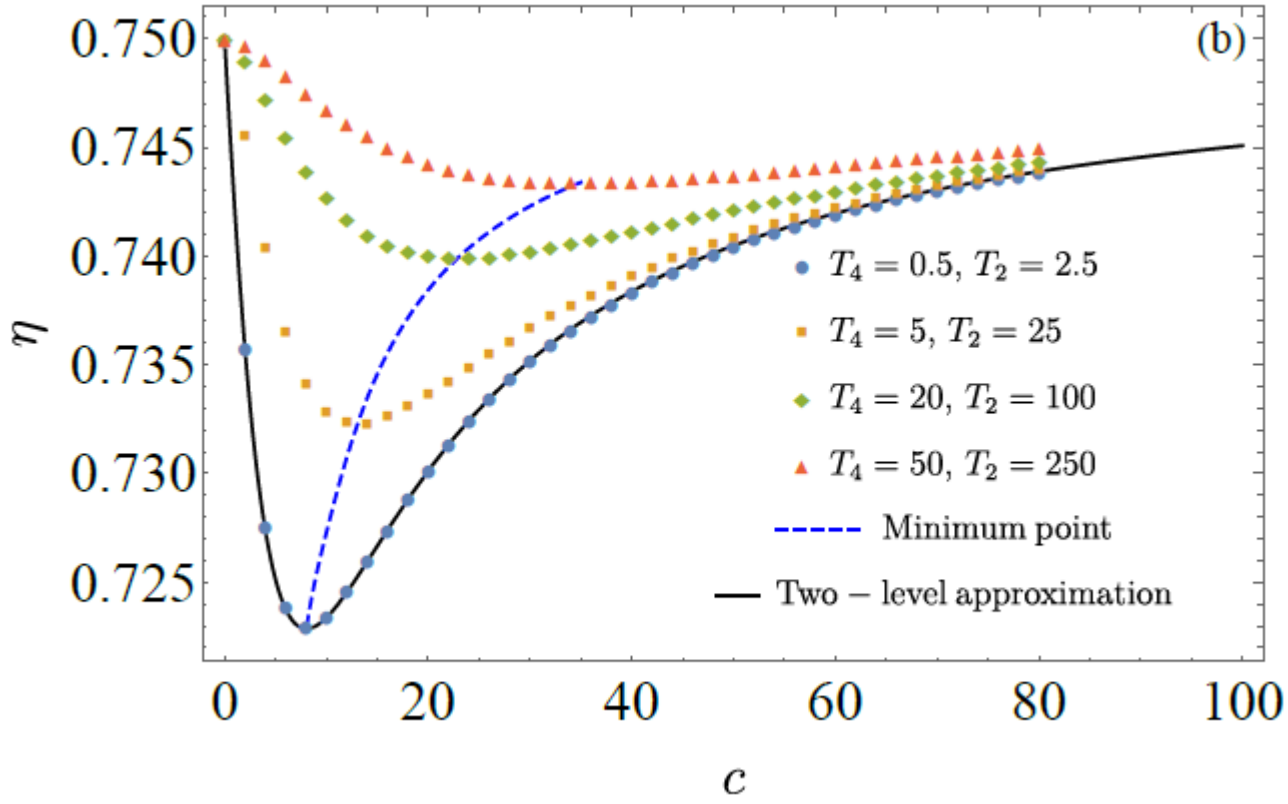
- The efficiency is rewritten as

$$\eta = \frac{\sum_{\{n_i\}} \left(p^{(2)}_{\{n_i\}} - p^{(4)}_{\{n_i\}}\right) E^{(L_1)}_{\{n_i\}} \lambda_{\{n_i\}}(L_1, L_2)}{\sum_{\{n_i\}} \left(p^{(2)}_{\{n_i\}} - p^{(4)}_{\{n_i\}}\right) E^{(L_1)}_{\{n_i\}}},$$



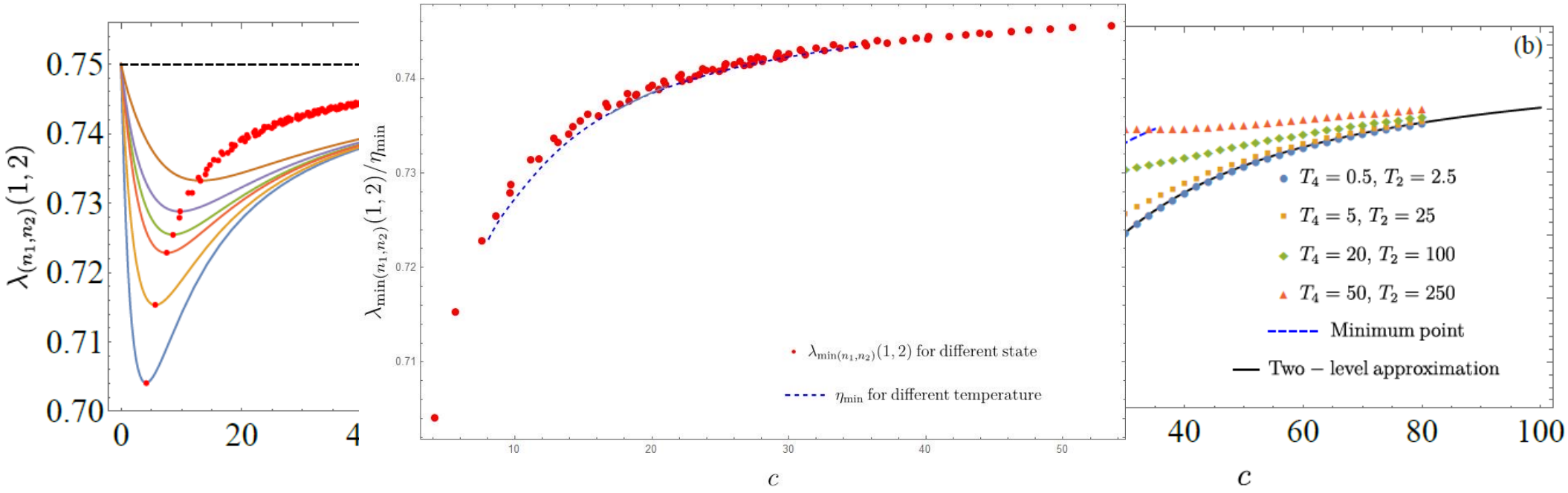
Interaction Effect on Efficiency

- For different temperature of the cold and hot bath

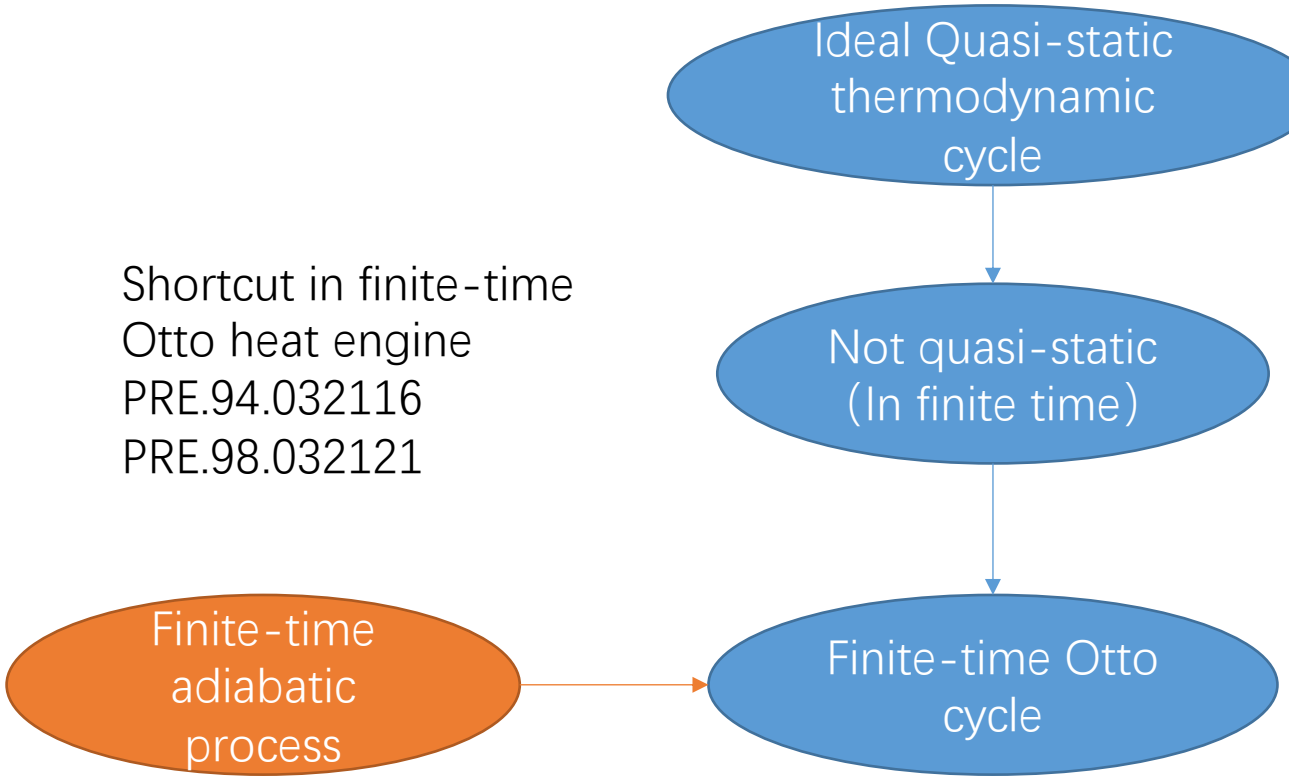


Interaction Effect on Efficiency

- $\lambda_{\min}$ matches with $\eta_{\min}$ at different temperature.



Finite-time Otto Heat Engine with Piston Model



Shortcut in finite-time Otto heat engine
PRE.94.032116
PRE.98.032121

Finite-time Carnot heat engine
Curzon-Ahlborn efficiency
F. Curzon and B. Ahlborn, Am. J. Phys. 43, 22 (1975) $\eta_{\mathrm{EMP}} = 1 - \sqrt{\frac{T_c}{T_h}}$

Low-dissipation heat engine
M. Esposito, et, al, PRL 105, 150603
Y.-H. Ma, D. Xu, H. Dong, and C.-P. Sun, Physical Review E 98, 042112

$1/\tau$ law for the irreversible entropy production for a finite-time isothermal process

We optimize the duration of the finite-time adiabatic process to gain high power.

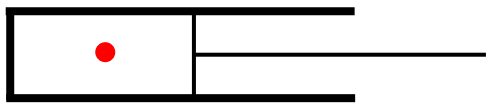
Finite-time Otto Heat Engine with Piston Model

Realize on piston model.

$$H(t) = \frac{\hat{p}^2}{2M} + V(x,t),$$

$$V(x,t) = \begin{cases} \infty & x < 0,\ x > L(t) \\ 0 & 0 \le x \le L(t) \end{cases}$$

$$L(t) = L_0 + (L_1 - L_0)\frac{t}{\tau}.$$



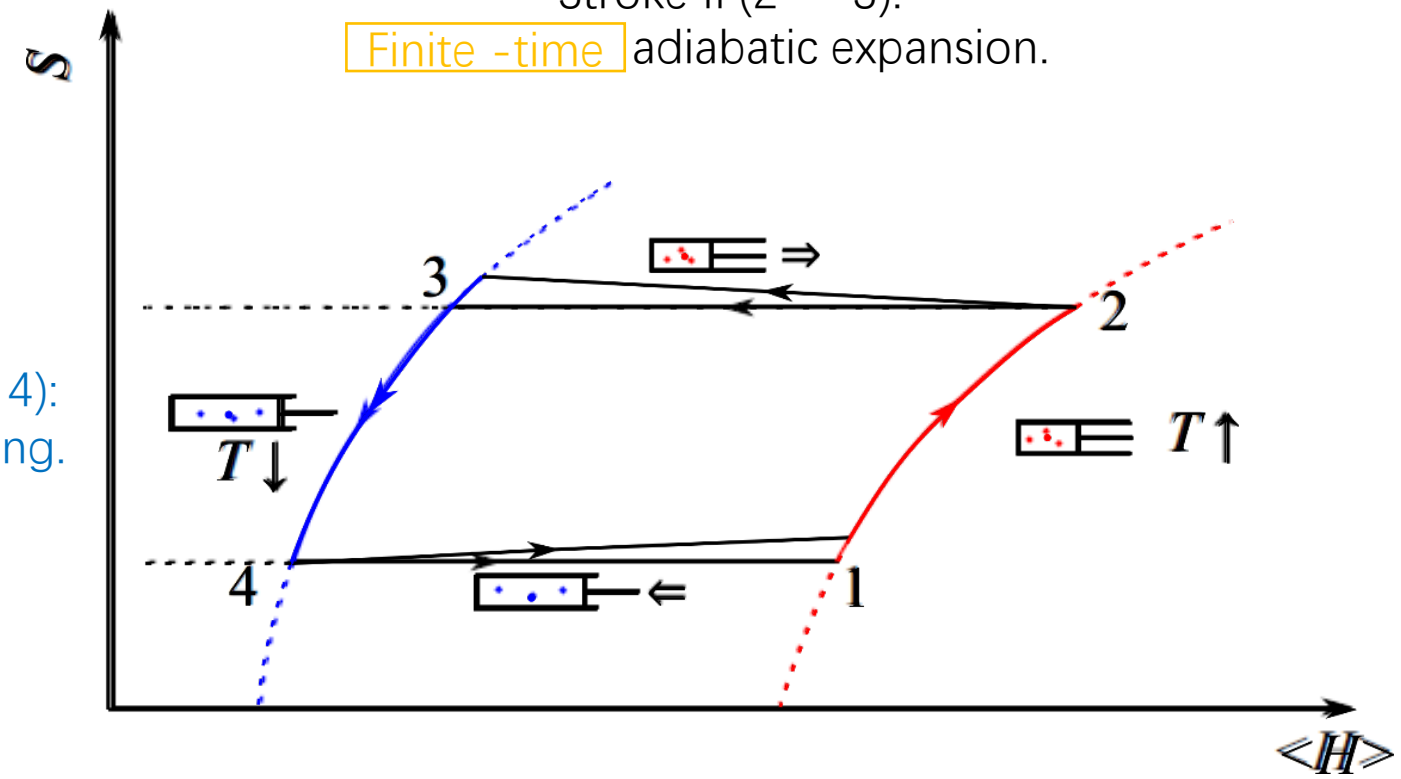
The exact solution exists
S. W. Doescher and M. H. Rice, American Journal of Physics 37, 1246

Validity of nonequilibrium work relations for the rapidly expanding quantum piston
H. T. Quan and C. Jarzynski, Physical Review E 85, 031102 (2012).

Finite-time Otto Heat Engine

For finite-time adiabatic processes II & IV, the evolution is determined by the Schrodinger equation

Stroke II (2 → 3):
Finite -time adiabatic expansion.



Stroke III (3 → 4):
Isochoric cooling.

Stroke I (1 → 2):
Isochoric heating.

Stroke IV (4 → 1):
Finite -time adiabatic compressing.

Finite-time Adiabatic Process

- The control Hamiltonian

$$H(t) = \sum_n E_n(t) \left|n(t)\right\rangle \left\langle n(t)\right| .$$

- The von Neumann equation

$$\dot{\rho} = -\frac{i}{\hbar} \left[H(t), \rho\right] .$$

- with a initial thermal state

$$\rho(0) = \sum_n p_n \left|n(0)\right\rangle \left\langle n(0)\right| .$$

- $\rho(t)$ follows as

$$\rho(t) = \sum_n p_n \left|\psi_n(t)\right\rangle \left\langle \psi_n(t)\right| .$$

Finite-time Adiabatic Process

- On the instantaneous eigenstate, we have

$$|\psi_n(t)\rangle = \sum_l c_{nl}(t)\,|j(t)\rangle\,.$$

- c_{nl} is obtained by

$$i\hbar\left(\dot{c}_{nl}(t) + \sum_j c_{nj}(t)\,\langle l(t)|\,\dot{j}(t)\rangle\right) = c_{nl}(t)E_l(t).$$

Finite-time Adiabatic Process

- The quantum adiabatic approximation gives $\left|\frac{\hbar\langle l(t)\,|\dot{n}(t)\rangle}{(E_n(t)-E_l(t))}\right| \ll 1$

$$c_{nn}^{(0)}(t) = \exp\left[\int_0^t -\frac{i}{\hbar}\left(E_n(t_1) - i\left\langle n(t_1)\right|\dot{n}(t_1)\right\rangle\right)dt_1\right], \qquad c_{nl}^{(0)}(t) = 0,\ n \neq l$$

- We solve to the next order

$$i\hbar\left(\dot{c}_{nl}^{(1)}(t) + \sum_j c_{nj}^{(0)}(t)\left\langle l(t)\right|\dot{j}(t)\right\rangle\right) = c_{nl}^{(1)}(t)E_l(t).$$

Finite-time Adiabatic Process

- The work for the finite-time adiabatic process

$$\begin{aligned} W(\tau) &= \mathrm{tr}\rho(\tau)H(\tau) - \mathrm{tr}\rho(0)H(0) \\ &= \sum_{nl} p_n \left(|c_{nl}(\tau)|^2 E_l(\tau) - E_n(0) \right) \end{aligned}$$

- The work for quasi-static process

$$W^{\infty} = \sum_{n} p_n \left(E_n(\tau) - E_n(0) \right)$$

- We define the extra work

$$\begin{aligned} W^{(\mathrm{ex})}(\tau) &= W(\tau) - W^{\infty}. \\ &= \sum_{nl} p_n \left| c_{nl}(\tau) \right|^2 \left(E_l(\tau) - E_n(\tau) \right) \end{aligned}$$

Finite-time Adiabatic Process

- We find $W^{(\mathrm{ex})}(\tau)$ decrease with oscillation with longer duration τ

$$W^{(\mathrm{ex})}(\tau) = \frac{\Sigma}{\tau^2} + \frac{\text{oscillated terms}}{\tau^2}$$

$W^{(\text{mean ex})}(\tau)$ $\qquad$ $W^{(\text{oscillated ex})}(\tau)$

- The oscillated term comes from the oscillation of $|c_{nl}(\tau)|^2$, $n \neq l$
- Higher-order quantum adiabatic theorem ⟹ Σ, oscillated term
- $W^{(\mathrm{ex})}(\tau) > 0$ for initial thermal state ⟹ positive Σ

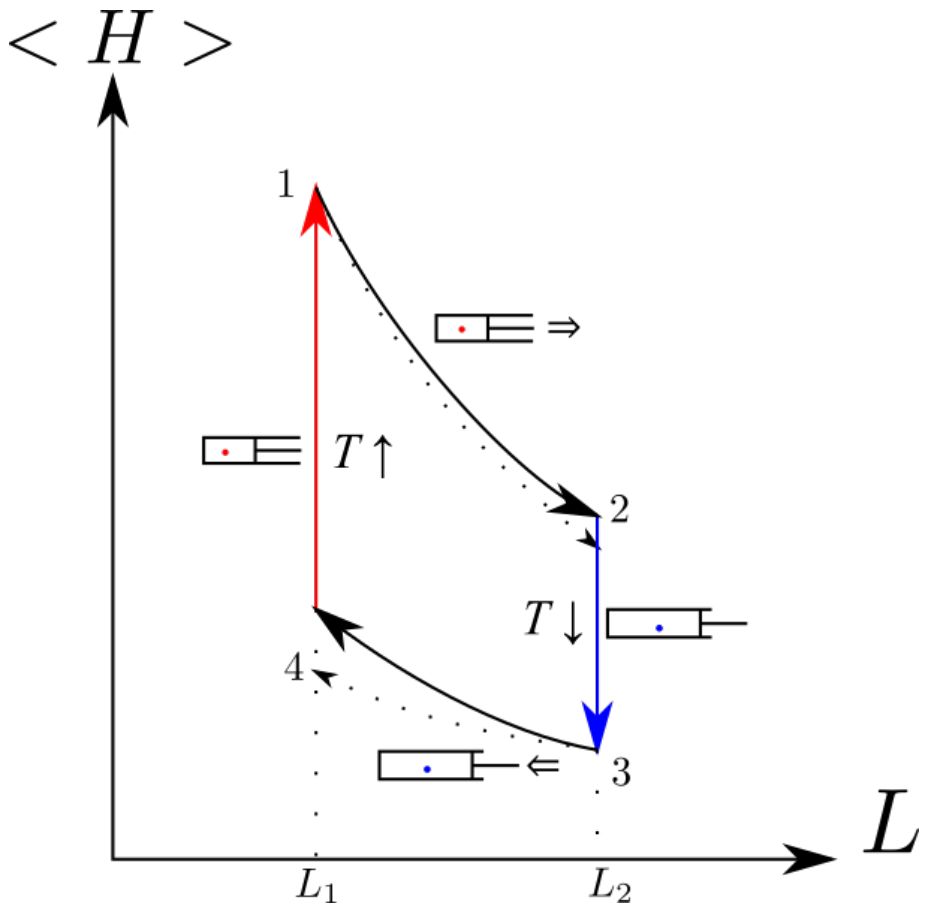
Finite-time Otto Heat Engine

$$W^{(\mathrm{ex})}(\tau) \propto \frac{1}{\tau^2} \longrightarrow \text{Maximum power}$$

- Neglect the time consuming of the isochoric process

$$\langle H\rangle_2 = \langle H\rangle_2^{\infty} + \frac{\Sigma_1}{\tau_1^2}, \qquad Q_h = \langle H\rangle_1 - \langle H\rangle_4 ,$$

$$\langle H\rangle_4 = \langle H\rangle_4^{\infty} + \frac{\Sigma_2}{\tau_2^2}, \qquad Q_c = \langle H\rangle_3 - \langle H\rangle_2 .$$

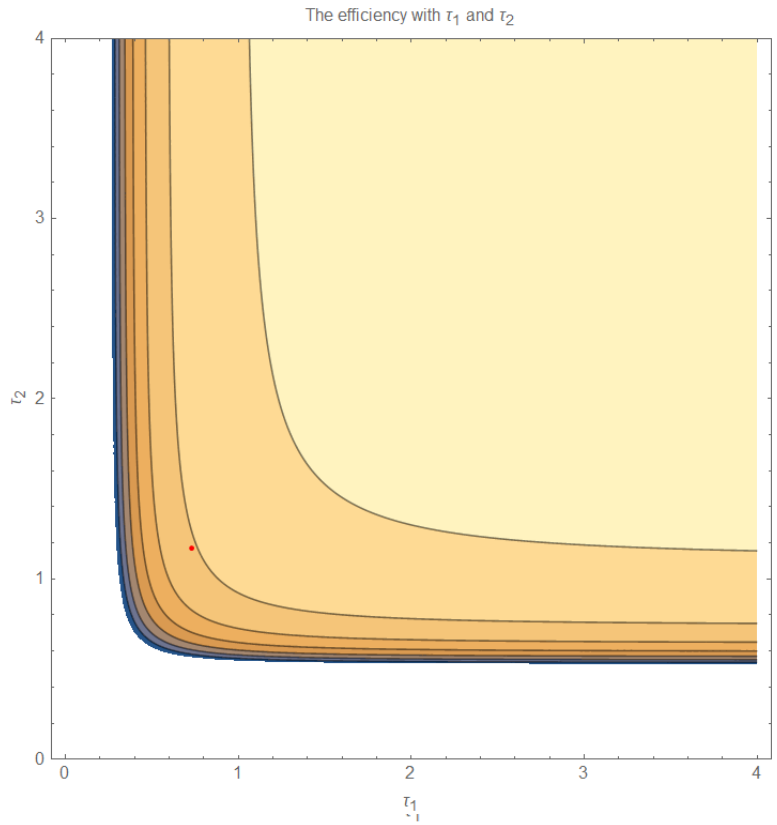


Finite-time Otto Heat Engine

- The power and the efficiency is

$$P = \frac{W_{\text{cycle}}^{\infty}}{\tau_1 + \tau_2} - \frac{1}{\tau_1 + \tau_2}\left(\frac{\Sigma_1}{\tau_1^2} + \frac{\Sigma_2}{\tau_2^2}\right),$$

$$\eta = \frac{W_{\text{cycle}}^{\infty} - \left(\frac{\Sigma_1}{\tau_1^2} + \frac{\Sigma_2}{\tau_2^2}\right)}{\frac{W_{\text{cycle}}^{\infty}}{\eta^{\infty}} - \frac{\Sigma_2}{\tau_2^2}}.$$



Finite-time Otto Heat Engine

- The optimal power is obtained by $\frac{\partial P}{\partial \tau_1} = 0,\ \frac{\partial P}{\partial \tau_2} = 0$

$$\begin{cases} \frac{\frac{\Sigma_1}{\tau_1^2} + \frac{\Sigma_2}{\tau_2^2}}{(\tau_1+\tau_2)^2} + \frac{2\Sigma_1}{\tau_1^3(\tau_1+\tau_2)} - \frac{W_{\mathrm{cycle}}^{\infty}}{(\tau_1+\tau_2)^2} = 0 \\ \frac{\frac{\Sigma_1}{\tau_1^2} + \frac{\Sigma_2}{\tau_2^2}}{(\tau_1+\tau_2)^2} + \frac{2\Sigma_2}{\tau_2^3(\tau_1+\tau_2)} - \frac{W_{\mathrm{cycle}}^{\infty}}{(\tau_1+\tau_2)^2} = 0 \end{cases}$$

Finite-time Otto Heat Engine

- The maximum power

$$P^* = \frac{2}{3\sqrt{3}} \left(\frac{W^{\infty}_{\rm cycle}}{\Sigma_1^{1/3} + \Sigma_2^{1/3}} \right)^{3/2}.$$

- The efficiency at maximum power

$$\eta_{\rm EMP} = \frac{2\eta^{\infty}}{3 - \frac{\Sigma_2^{1/3}}{\Sigma_1^{1/3} + \Sigma_2^{1/3}} \eta^{\infty}},$$

$$W^{\infty}_{\rm cycle} = \langle H \rangle_1 - \langle H \rangle_4^{\infty} + \langle H \rangle_3 - \langle H \rangle_2^{\infty}$$

- satisfying

$$\frac{2\eta^{\infty}}{3} \leq \eta_{\rm EMP} \leq \frac{2\eta^{\infty}}{3 - \eta^{\infty}}.$$

Finite-time Otto Heat Engine

- The optimal duration and the work for quasi-static cycle

$$\tau_1^* = \sqrt{3}\sqrt{\frac{\Sigma_1^{2/3}\Sigma_2^{1/3} + \Sigma_1}{W_{\rm cycle}^{\infty}}} \qquad \tau_2^* = \sqrt{3}\sqrt{\frac{\Sigma_2^{2/3}\Sigma_1^{1/3} + \Sigma_2}{W_{\rm cycle}^{\infty}}}$$

What is Σ_1, Σ_2 microscopically?

Realization on Piston Model

- We consider a single particle in a box with time-dependent length

$$H(t) = \frac{\hat{p}^2}{2M} + V(x,t),$$

$$V(x,t) = \begin{cases} \infty & x < 0,\ x > L(t) \\ 0 & 0 \le x \le L(t) \end{cases}$$

- The protocol is

$$L(t) = L_0 + (L_1 - L_0)\frac{t}{\tau}.$$

Realization on Piston Model

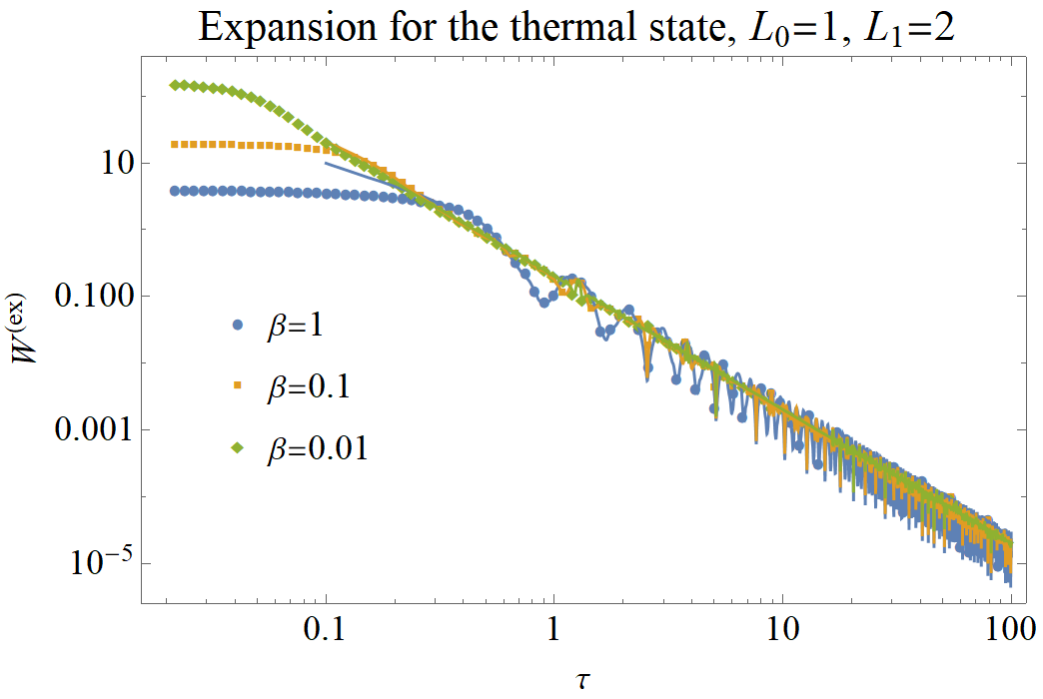
- At quantum adiabatic limit, the mean part and the oscillation part are

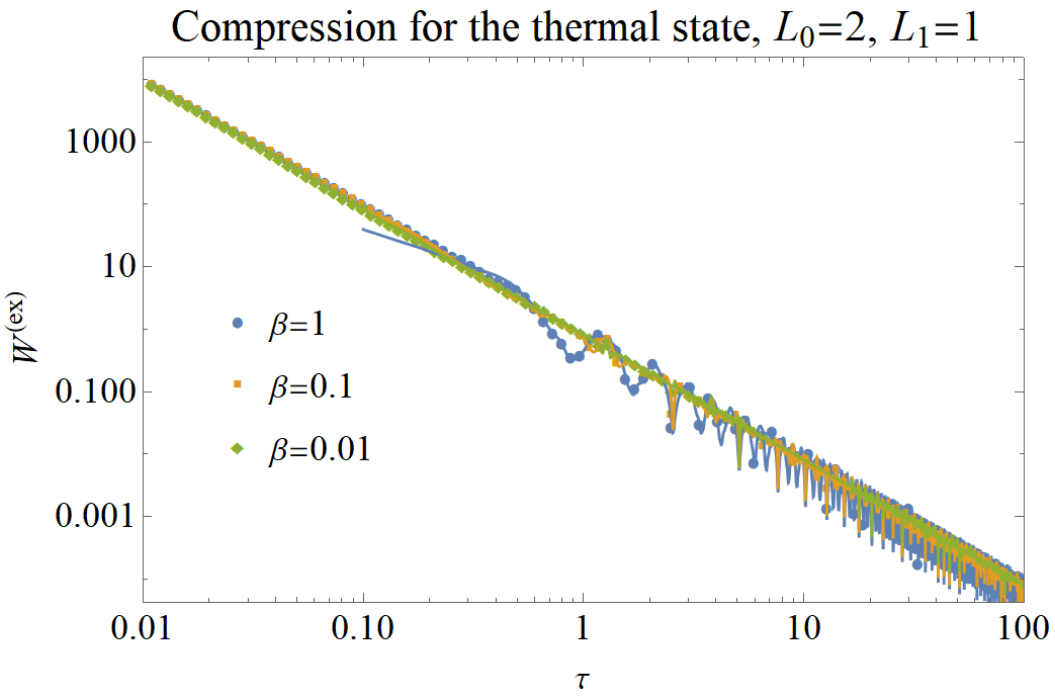
$$W_\beta^{(\text{ex mean})}(\tau) = \sum_{n=0}^{\infty} p_n \frac{M}{\tau^2} \frac{(L_1 - L_0)^2 \left(L_1^2 + L_0^2\right)}{L_1^2} \left(\frac{1}{6} - \frac{1}{4\pi^2 n^2}\right)$$

$$W_\beta^{(\text{ex oscillation})}(\tau) = -\sum_{n=0}^{\infty} p_n \frac{M}{\tau^2} \frac{16 L_0 \left(L_1 - L_0\right)^2}{L_1} \sum_{l \neq n} \frac{l^2 n^2}{\pi^2 (l^2 - n^2)^3} \cos\left(\frac{(n^2 - l^2)\pi^2 \hbar}{2 M L_0} \frac{\tau}{L_1}\right),$$

Realization on Piston Model

- Matches with exact numerical result for large τ.





Realization on Piston Model

- The corresponding coefficient for high temperature is

$$\Sigma_1 = m\frac{(L_1 - L_0)^2\left(L_1^2 + L_0^2\right)}{6L_1^2}$$
$$\Sigma_2 = m\frac{(L_1 - L_0)^2\left(L_1^2 + L_0^2\right)}{6L_0^2}.$$

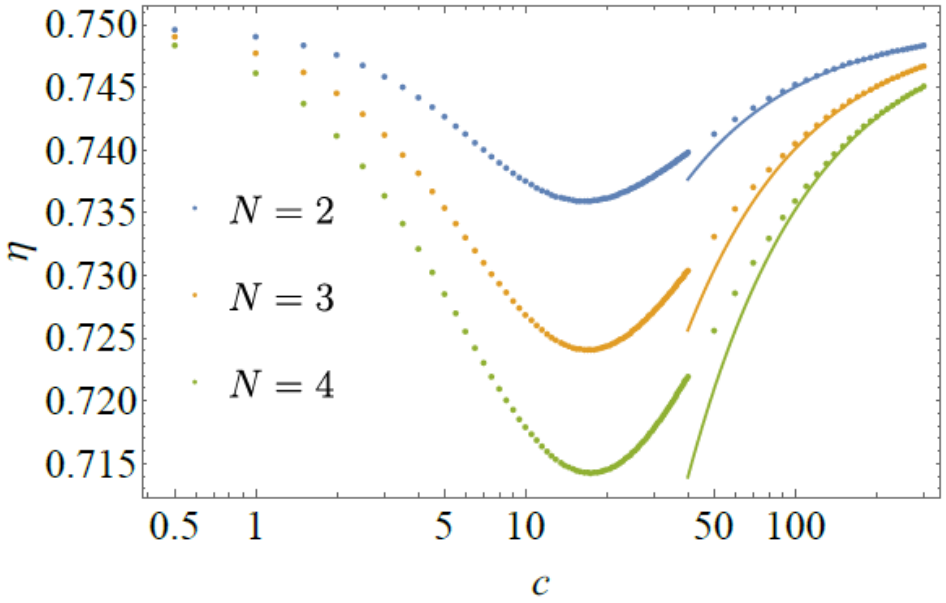
- The optimal power and the corresponding efficiency is

$$P^* = \frac{2\sqrt{2}}{3}\left(\frac{k_B\left(T_2L_0^2 - T_1L_1^2\right)\left(\frac{1}{L_0^2} - \frac{1}{L_1^2}\right)}{\left[m(L_1 - L_0)^2\left(L_1^2 + L_0^2\right)\right]^{1/3}\left(\frac{1}{L_1^{2/3}} + \frac{1}{L_0^{2/3}}\right)}\right)^{3/2}$$
$$\eta_{\mathrm{EMP}} = \frac{2\eta^{\infty}}{3 - \frac{1}{\left(\frac{L_0}{L_1}\right)^{2/3}+1}\eta^{\infty}}.$$

Summary

- Bose-Fermi Duality

$$\eta = 1 - \left(\frac{L_1 + \frac{2(N-1)}{c}}{L_2 + \frac{2(N-1)}{c}} \right)^2$$



- For finite-time adiabatic process, $W^{(\mathrm{ex})}(\tau) \propto \frac{1}{\tau^2} > 0$;

- The EMP for a finite-time Otto cycle $\eta_{\mathrm{EMP}} = \frac{2\eta^{\infty}}{3 - \frac{\Sigma_2^{1/3}}{\Sigma_1^{1/3}+\Sigma_2^{1/3}}\eta^{\infty}}$.


Thank You!